

Quantitative Analysis For Management Decisions (Q.T)

Unit:1

- Introduction to quantitative techniques
- * Functions and applications to business
- * Limits of function and continuity

Unit:2 Concept of derivatives, techniques of differentiation, Maxima & Minima.

- * Integration and its applications in Managerial decision.

Unit:3 Progression.

- * Arithmetic and Geometric progressions and their applications.
- * Annuities and Present Value.

Unit:4 Vectors and Matrices

- * Geometrical and physical interpretation of Vectors
- * Introduction to Matrices
- * Multiplication of Matrices
- * Inverse of a Matrix
- * Rank of a matrix and matrix application in management.

Unit:5 Gamification.

- * Games, Characteristics theory and rules.
- * Game elements
 - Two person, zero sum games
 - Methods with and without saddle point
 - Dominance Rule & Mixed strategies.

UNIT-1

CHAPTER-1 Introduction to Quantitative techniques

Definition: Collection of mathematical and numerical methods used in the solution of numerical decision making problem is known as quantitative methods or quantitative techniques.

Classification of Quantitative techniques

Quantitative techniques are broadly classified into two types, they are

- * Statistical techniques

- * Programming.

Statistical Techniques.

- * Statistical techniques are those techniques which are used in conducting the statistical enquiry consulting phenomena.

There are six methods in statistical techniques.

- * Probability theory and Sampling Analysis

- * Time Series Analysis

- * Co-relation and Regression Analysis

- * Interpolation and Extrapolation methods.

- * Ratio Analysis

- * Statistical Quality Control.

Probability and Sampling Analysis

Sampling data is selected and statistical data will be applied on a selected data again.

They are classified into two types.

- * Probability or Random Sampling

- * Non-Probability or Procedure Simplifier.

* Probability is constructed that every element from the total population has equal probability to be selected.

Co-relation & Regression Analysis

Co-relation analysis measures the closeness of a relation. Regression analysis is used to study the degree of a functional relations among two or more variables.

* The Index number of statistical technique will be measured functional in price Economic activities and time period related to the base.

Time Series Analysis

Using the statistical techniques the series of time period analysis to know the function of trend.

Interpolation & Extrapolation.

The method of finding intermediate values (or) missing values within the given range in the problem statement is known as Interpolation.

* If the estimated values lies outside the two extreme points in given problem statement known as Extrapolation.

* Ratio Analysis

This statistical techniques is used for analyse the financial status or any business industrial constructions to take a credit decision by evaluating the probability in terms of ratio to find the position of pay is known as Ratio Analysis.

Statistical Quality Control:

It is used to control the quality of goods by the manufacturing industries with the help of pre-defined quality measures to obtain the best results.

Programming Techniques:

These techniques are modern techniques used by the decision maker in the present trend. They are classified as

- * Linear programming techniques

- * Decision theory

- * Game theory

- * Stimulation

- * Management

- * Innovative planning

- * Network Analysis

- * Integrated production model techniques.

Linear Programming Techniques:

These techniques used to allocate scarce resources in an optimal manner in problems for scheduling products and some other entities in different modules.

Decision theory.

The decision theory enables business managers to select the best sources of an action and information given in probability from decision theory techniques. Executor judgement can be systematically done throughout the problem analysis.

Game theory:

This is the physical Entity used to determine the optimal strategy in a competitive situation and enables to maximize the profit and minimize the losses by opting the optimal strategies.

Stimulation.

It is used to estimate and operation process of actual performance in significance of stimulation highlights in the facts i.e., able to find the alternative sources of action in any uncertain situation where mathematical formation is not possible.

Management.

This theory helps us management in minimize the waiting and servicing cost and also analyses the feasibility of adding facilities and helps to take the profitable decisions.

Innovative Planning

The main objective of innovative planning is to achieve balance between the cost of holding stock and the benefits from existing stock which enables the decision maker to take the proper decision when to buy and how to manage the buyer cost.

Network Analysis

Project Evaluation and revenue technique

Programming evaluation enables to determine the earliest item for each event activities helps in identification and clarification at critical path while developing and managing resources from one activity from another.

Integrated production model technique

It is used to minimize the cost with respect to work force, production & Innovation.

Properties of Limits

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

$$\lim_{x \rightarrow a} \sin f(x) = \sin \left[\lim_{x \rightarrow a} f(x) \right]$$

$$\lim_{x \rightarrow a} \log f(x) = \log \left[\lim_{x \rightarrow a} f(x) \right]$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$$

$$\lim_{x \rightarrow a} \frac{e^x - 1}{x} = 1$$

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$$

1) Evaluate $\lim_{x \rightarrow 2}$ in the given function $\frac{x^2 - 5x + 6}{x^2 + x - 6}$

$$\text{Given } \lim_{x \rightarrow 2} \left[\frac{x^2 - 5x + 6}{x^2 + x - 6} \right]$$

$$= \frac{(2)^2 - 5(2) + 6}{2^2 + 2 - 6}$$

$$= \frac{4 - 10 + 6}{4 + 2 - 6} = 0/0 = 0 \text{ (Indeterminate form)}$$

It is indetermination form, so we need to find the roots for given function.

from the given Equation find out the roots for both numerator & denominator.

$$\text{Numerator } x^2 - 5x + 6$$

$$x^2 - 2x - 3x + 6$$

$$x(x-2) - 3(x-2)$$

$$(x-3)(x-2)$$

$$\text{Denominator } x^2 + x - 6$$

$$x^2 + 3x - 2x - 6$$

$$x(x+3) - 2(x+3)$$

$$(x-2)(x+3)$$

Substitute the roots of numerator & denominator in an Equation.

$$\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 + x - 6}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{(x-2)(x+3)}$$

$$\lim_{x \rightarrow 2} \frac{x-3}{x+3} =$$

$$= \frac{2-3}{2+3}$$

$$= -1/5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = -1/5$$

2) Evaluate $\lim_{x \rightarrow 2} \frac{x^2 + 5x + 6}{2x^2 - 3x}$

Given $\lim_{x \rightarrow 2} \frac{x^2 + 5x + 6}{2x^2 - 3x}$

$$= \frac{2^2 + 5(2) + 6}{2(2)^2 - 3(2)}$$

$$= \frac{4 + 10 + 6}{2(4) - 6} = \frac{20}{2} = 10$$

$$\lim_{x \rightarrow 2} \frac{x^2 + 5x + 6}{2x^2 - 3x} = 10$$

3) Evaluate $\lim_{x \rightarrow 3} \frac{x+2}{3+2}$

Given $\lim_{x \rightarrow 3} \frac{x+2}{3+2}$

$$= 5$$

4) Evaluate $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

Given Equation $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

$$= \frac{4^2 - 16}{4 - 4}$$

$$= \frac{16 - 16}{0}$$

$$= 0/0 = 0 \text{ (Indeterminate form)}$$

It is indetermination form, so we need to find the roots for given function.

Numerator is in the form of $a^2 - b^2$

$$\text{i.e. } a^2 - b^2 = (a+b)(a-b)$$

$$\text{Numerator } x^2 - 16 = (x+4)(x-4)$$

Substitute the numerator in the given function.

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{(x+4)(x-4)}{(x-4)}$$

$$\begin{aligned} \lim_{x \rightarrow 4} (x+4) \\ = 4+4 \\ = 8 \end{aligned}$$

$$\lim_{x \rightarrow 2} \frac{x^2 - 16}{x - 4} = 8$$

$$5) \lim_{x \rightarrow 2} \frac{(x+4)(2x+1)}{x^2 + 5x + 2}$$

$$\text{Given } \lim_{x \rightarrow 2} \frac{(x+4)(2x+1)}{x^2 + 5x + 2}$$

$$\frac{(2+4)(2(2)+1)}{2^2 + 5(2) + 2}$$

$$2^2 + 5(2) + 2$$

$$\frac{6 \times 5}{4 + 10 + 2}$$

$$4 + 10 + 2$$

$$= \frac{30}{16} = 15/8$$

$$\lim_{x \rightarrow 2} \frac{(x+4)(2x+1)}{x^2 + 5x + 2} = 15/8$$

$$6) \lim_{x \rightarrow 1} x^2 + 2x + 3$$

$$\text{Given } \lim_{x \rightarrow 1} x^2 + 2x + 3$$

$$(1)^2 + 2(1) + 3$$

$$1 + 2 + 3$$

$$= 6$$

$$7) \lim_{x \rightarrow 0} \frac{1}{x^2 - 3x + 2}$$

$$\frac{1}{0^2 - 3(0) + 2}$$

$$= \frac{1}{0 - 0 + 2} = 1/2$$

$$8) \lim_{x \rightarrow \infty} \frac{3x^2 - 5x + 9}{2x^2 + 4x + 7}$$

If we substitute ' ∞ ' in place of ' x ' then we get indeterminate form.

So, we need to convert ∞ to 0 to get determinate

$$\text{if } x = \infty$$

$$1/x = 1/\infty$$

$$1/x = 0$$

$$\lim_{1/x \rightarrow 0} \frac{3x^2 - 5x + 9}{2x^2 + 4x + 7}$$

Divide the above function with highest degree ' x^2 '

$$\lim_{1/x \rightarrow 0} \frac{\frac{3x^2 - 5x + 9}{x^2}}{\frac{2x^2 + 4x + 7}{x^2}}$$

$$\lim_{1/x \rightarrow 0} \frac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{9}{x^2}}{\frac{2x^2}{x^2} + \frac{4x}{x^2} + \frac{7}{x^2}}$$

$$\lim_{1/x \rightarrow 0} \frac{3 - 5/x + 9/x^2}{2 + 4/x + 7/x^2}$$

$$\lim_{1/x \rightarrow 0} \frac{3 - 5(1/x) + 9(1/x)^2}{2 + 4(1/x) + 7(1/x)^2}$$

$$\frac{3 - 5(0) + 9(0)}{2 + 4(0) + 7(0)}$$

$$= \frac{3 - 0 + 0}{2 + 0 + 0} = \frac{3}{2}$$

$$\therefore \lim_{x \rightarrow \infty} \frac{3x^2 - 5x + 9}{2x^2 + 4x + 7} = \frac{3}{2}$$

$$9) \lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 + 5x - 7} - \sqrt{4x^2 + 7x + 5}}{2x^2 - 3}$$

If we substitute ' ∞ ' in the place of ' x ' in the given function then it will give indeterminate form

So, we need to convert ' ∞ ' to 0 to get the determinate.

$$\text{if } x = \infty$$

$$1/x = 1/\infty$$

$$1/x = 0$$

$$\text{Now, } \lim_{1/x \rightarrow 0} \frac{\sqrt{9x^2 + 5x - 7} - \sqrt{4x^2 + 7x + 5}}{2x^2 - 3}$$

Now, Divide the given function with x^2

$$\lim_{1/x \rightarrow 0} \frac{\frac{\sqrt{9x^2 + 5x - 7} - \sqrt{4x^2 + 7x + 5}}{x^2}}{\frac{2x^2 - 3}{x^2}}$$

$$\lim_{1/x \rightarrow 0} \frac{\sqrt{\frac{9x^2}{x^2} + \frac{5x}{x^2} - \frac{7}{x^2}} - \sqrt{\frac{4x^2}{x^2} + \frac{7x}{x^2} + \frac{5}{x^2}}}{\frac{2x^2}{x^2} - \frac{3}{x^2}}$$

$$\lim_{1/x \rightarrow 0} \frac{\sqrt{9 + 5/x - 7/x^2} - \sqrt{4 + 7/x + 5/x^2}}{2 - 3/x^2}$$

$$\lim_{1/x \rightarrow 0} \frac{\sqrt{9 + 5(1/x) - 7(1/x)^2} - \sqrt{4 + 7(1/x) + 5(1/x)^2}}{2 - 3(1/x)^2}$$

$$\frac{\sqrt{9 + 5(0) - 7(0)} - \sqrt{4 + 7(0) + 5(0)}}{2 - 3(0)} = \frac{\sqrt{9} - \sqrt{4}}{2} = \frac{3 - 2}{2} = \frac{1}{2}$$

Limits and Continuity

The limit of a function $f(x)$ as extended to any arbitrary chosen positive number. However, small but non-zero there exists corresponding positive number that represents with infinity.

such that, $|f(x) - l| < \epsilon \forall x$, such that $0 < |f(x) - l| < \epsilon$

Evaluation Limits

Case:1 To Evaluate $\lim_{x \rightarrow a} f(x)$ first we substitute $x=a$ and $x=a$ and we observe, whether the expression takes determination form or not (indetermination).

If the expression does not takes in indetermination form then the values obtain itself is the limit of the given function.

If the expression gives indetermination form then we perform factorisation method to find the limits for the given function.

Case:2 To Evaluate $\lim_{x \rightarrow \infty} f(x)$, first we need to convert ' ∞ ' to ' 0 ' form. To convert ' ∞ ' to ' 0 ' we can perform $1/x$ (inverse of ∞).

Now, the given function becomes $\lim_{1/x \rightarrow 0} f(x)$

Now divide the numerator and denominator with highest degree variables in the given function.

After Evaluating now we substitute in the place of $1/x$. Now obtained values is the limit of the given function.

Any Equation contains $+$, $-$, $\times$, $\div$, f with highest degree more than 2 - polynomial

* The Equation which has trigonometry & logarithms - transcendental

* Equation in the form of a/b - Rational

Functions & Applications

Polynomial function

A function is of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_nx^0$ is called polynomial function of Equations
Eg: $3x^4 + 4x^3 + 3x^2 + x + 4 = 0$

* A polynomial function of Equations are the combinations of constant values, variable & some mathematical operations like $(+, -, \div, \times, \sqrt{\quad})$ is set to be a polynomial

Rational Equations

A function $f(x)$ defined by quality of two polynomials is called Rationalization function.

Ex: $y = \frac{4x^2 + x + 1}{x + 6}$

Irrationalisation function

A function of the form $y = \sqrt{x+1}$ is called irrationalised function.

Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x}$

Given Equation $\lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x}$

$$\frac{\sqrt{a+x} - \sqrt{a-x}}{x}$$

$\frac{0}{0}$

$$\frac{a-x}{0} = 0/0 = 0 \text{ (indeterminate form)}$$

It is an indeterminate form

So, we need to convert the given function from indeterminate to determinate.

for conversion we need to do the rationalisation of the given function.

for Rationalisation, multiply both numerator and denominator with ' $\sqrt{a+x} + \sqrt{a-x}$ '

$$\lim_{x \rightarrow 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$$

Numerator is in the form of $(a-b)(a+b) = a^2 - b^2$

$$\lim_{x \rightarrow 0} \frac{(\sqrt{a+x})^2 - (\sqrt{a-x})^2}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$\lim_{x \rightarrow 0} \frac{(a+x) - (a-x)}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$\lim_{x \rightarrow 0} \frac{a+x-a+x}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$\lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$\lim_{x \rightarrow 0} \frac{2}{\sqrt{a+x} + \sqrt{a-x}} = \frac{2}{\sqrt{a+0} + \sqrt{a-0}} = \frac{2}{2\sqrt{a}} = \frac{1}{\sqrt{a}}$$

$$2) \lim_{x \rightarrow 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{x}$$

Given function $\lim_{x \rightarrow 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{x}$

$$\lim_{x \rightarrow 0} \frac{\sqrt{5+0} - \sqrt{5-0}}{0}$$

$$\frac{\sqrt{5}-\sqrt{5}}{0} = 0/0 = 0 \text{ (indeterminate form)}$$

It is an indeterminate form. So, we need to convert indeterminate form to determinate form. For conversion we need to do rationalise the given function.

For Rationalisation multiply both numerator and denominator with $\sqrt{5+x} + \sqrt{5-x}$

$$\lim_{x \rightarrow 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{x} \times \frac{\sqrt{5+x} + \sqrt{5-x}}{\sqrt{5+x} + \sqrt{5-x}}$$

Numerator is in the form of $(a-b)(a+b) = a^2 - b^2$

$$\lim_{x \rightarrow 0} \frac{(\sqrt{5+x})^2 - (\sqrt{5-x})^2}{x(\sqrt{5+x} + \sqrt{5-x})}$$

$$\lim_{x \rightarrow 0} \frac{5+x - (5-x)}{x(\sqrt{5+x} + \sqrt{5-x})}$$

$$\lim_{x \rightarrow 0} \frac{5+x-5+x}{x(\sqrt{5+x} + \sqrt{5-x})}$$

$$\lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{5+x} + \sqrt{5-x})}$$

$$\frac{2}{\sqrt{5+0} + \sqrt{5-0}} = \frac{2}{\sqrt{5} + \sqrt{5}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$3) \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$$

Given

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$$

$$\frac{0}{\sqrt{1+0} - \sqrt{1-0}} = \frac{0}{\sqrt{1} - \sqrt{1}} = \frac{0}{0} = 0 \text{ (indeterminate form)}$$

It is an indeterminate form. So, we need to convert indeterminate form to determinate

for conversion we need to do rationalise the given function.

for Rationalisation multiply both numerator and denominator with $\sqrt{1+x} + \sqrt{1-x}$

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

Denominator is in the form $(a-b)(a+b) = a^2 - b^2$

$$\lim_{x \rightarrow 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}$$

$$\lim_{x \rightarrow 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{(1+x) - (1-x)}$$

$$\lim_{x \rightarrow 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{x+x}$$

$$\lim_{x \rightarrow 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{2x}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{2}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+0} + \sqrt{1-0}}{2}$$

$$\frac{2\sqrt{1}}{2} = \sqrt{1} = 1$$

$$4) \lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{5x^2 - 4x - 1}$$

Given function $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{5x^2 - 4x - 1}$

$$\frac{(1)^2 - 3(1) + 2}{5(1)^2 - 4(1) - 1}$$

$$\frac{1 - 3 + 2}{5 - 4 - 1}$$

$$= \frac{0}{0} = 0/0 = 0 \text{ (Indeterminate form)}$$

It is an indeterminate form. So we need to convert indeterminate form determinate form.

for conversion we need to find the roots of both numerator and denominator.

Numerator: $x^2 - 3x + 2$

$$x^2 - 2x - x + 2$$

$$x(x-2) - 1(x-2)$$

$$(x-1)(x-2)$$

Denominator: $5x^2 - 4x - 1$

$$5x^2 - 5x + x - 1$$

$$5x(x-1) + 1(x-1)$$

$$(5x+1)(x-1)$$

substitute the factors of numerator & denominator in the given function

$$\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{5x^2 - 4x - 1}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)(x-2)}{(5x+1)(x-1)}$$

$$\lim_{x \rightarrow 1} \frac{(x-2)}{(5x+1)}$$

$$\lim_{x \rightarrow 1} \frac{(1-2)}{(5(1)+1)}$$

$$\frac{1-2}{5(1)+1} = -1/6$$

Continuous function

A function is said to be continuous function if it satisfies the $R.H.S = L.H.S$

$$\Rightarrow f(x) = \begin{cases} x^y & \text{if } x \leq 1 \\ x & \text{if } x \geq 1 \end{cases}$$

Q.1 Given function $f(x) = \begin{cases} x^y & \text{if } x \leq 1 \\ x & \text{if } x \geq 1 \end{cases}$

from the given function $f(x)$ the limit is '1'.

$$\begin{array}{ll} \text{L.H.S} & \text{R.H.S} \\ \lim_{x \rightarrow 1} x^y = 1^y = 1 & \lim_{x \rightarrow 1} x = 1 \end{array}$$

$$L.H.S = R.H.S$$

$\therefore L.H.S = R.H.S$, then the given function is continuous.

Q.2 Check whether the given function is continuous or not

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & , \text{ if } 0 < x < 2 \\ 2 - 8x^{-3} & , \text{ if } x > 2 \end{cases}$$

Given function $f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & , \text{ if } 0 < x < 2 \\ 2 - 8x^{-3} & , \text{ if } x > 2 \end{cases}$

from the given function $f(x)$, the limit is 2

$$\text{L.H.S } \frac{1}{2}$$

$$\lim_{x \rightarrow 2} \frac{1}{2}(x^2 - 4)$$

$$\frac{1}{2}(2^2 - 4)$$

$$\frac{1}{2}(4 - 4)$$

$$\frac{1}{2}(0)$$

$$= 0$$

$$\text{R.H.S}$$

$$\lim_{x \rightarrow 2} 2 - 8x^{-3}$$

$$2 - \frac{8}{x^3}$$

$$2 - \frac{8}{2^3}$$

$$2 - \frac{8}{8}$$

$$2 - 1$$

$$= 1$$

$$= 1$$

$$3) f(x) = \begin{cases} x/2 & \text{if } x < 2 \\ x^{2/3} & \text{if } x \geq 2 \end{cases}$$

Given function $f(x) = \begin{cases} x/2 & \text{if } x < 2 \\ x^{2/3} & \text{if } x \geq 2 \end{cases}$

From the given function $f(x)$ the limit is 2

L.H.S

$$\lim_{x \rightarrow 2^-} x/2$$

$$= 2/2$$

$$= 1$$

R.H.S

$$\lim_{x \rightarrow 2^+} x^{2/3}$$

$$= 2^{2/3}$$

L.H.S $\neq$ R.H.S

The Given function is not Continuous.

$$4) f(x) = \begin{cases} x^2 & 1 \leq x \leq 2 \\ x-3 & x \geq 2 \end{cases}$$

Given function $f(x) = \begin{cases} x^2 & 1 \leq x \leq 2 \\ x-3 & x \geq 2 \end{cases}$

Case: 1 If the limit of the function is 1, 1

L.H.S

$$\lim_{x \rightarrow 1^-} x^2$$

$$(1)^2$$

$$= 1$$

R.H.S

$$\lim_{x \rightarrow 1^+} x^2$$

$$1^2 = 1$$

L.H.S = R.H.S

Case: 2 If the limit of the function is 2, 1

L.H.S

$$\lim_{x \rightarrow 2^-} x^2$$

$$2^2$$

$$= 4$$

R.H.S

$$\lim_{x \rightarrow 2^+} x-3$$

$$2-3$$

$$= -1$$

L.H.S $\neq$ R.H.S

$\therefore$ Given function is Continuous when limit is 1.

$$f(x) = \begin{cases} 4-x^2 & \text{if } x \leq 0 \\ x-5 & \text{if } 0 < x \leq 1 \\ 4x^2-9 & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \geq 2 \end{cases}$$

Given function $f(x) = \begin{cases} 4-x^2 & \text{if } x \leq 0 \\ x-5 & \text{if } 0 < x \leq 1 \\ 4x^2-9 & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \geq 2 \end{cases}$

from the given function the limit ranges are
0, 1, 2

Case: 1 when the limit of the function is 0

$$\text{L.H.S } \lim_{x \rightarrow 0} 4-x^2$$

$$4-0^2$$

$$= 4$$

$$\text{R.H.S}$$

$$\lim_{x \rightarrow 0} x-5$$

$$0-5$$

$$= -5$$

$$\text{L.H.S} \neq \text{R.H.S}$$

Case: 2 when the limit of the function is 1

$$\text{L.H.S } \lim_{x \rightarrow 1} x-5$$

$$1-5$$

$$= -4$$

$$\text{R.H.S } \lim_{x \rightarrow 1} 4x^2-9$$

$$4(1)^2-9$$

$$4-9$$

$$= -5$$

$$\text{L.H.S} \neq \text{R.H.S}$$

Case: 3 when the limit of the function is 2

$$\text{L.H.S } \lim_{x \rightarrow 2} 4x^2-9$$

$$4(2)^2-9$$

$$4(4)-9$$

$$16-9$$

$$= 7$$

$$\text{R.H.S } \lim_{x \rightarrow 2} 3x+4$$

$$3(2)+4$$

$$6+4$$

$$= 10$$

$$\text{L.H.S} \neq \text{R.H.S}$$

$\therefore$ Given function is not continuous in all the cases.

Previous year Question

Solve $\lim_{x \rightarrow -\infty} \frac{3x^y - 1}{2 - 3x - 4x^y}$

Given function $\lim_{x \rightarrow -\infty} \frac{3x^y - 1}{2 - 3x - 4x^y}$

If we substitute ' $-\infty$ ' in the given function then we will get indeterminate form.

So, we need to convert ' ∞ ' to '0'.

$$x = -\infty$$

$$x = 1/\infty$$

$$x = 0$$

$$\lim_{x \rightarrow 0} \frac{3x^y - 1}{2 - 3x - 4x^y}$$

$$\frac{3(0)^y - 1}{2 - 3(0) - 4(0)^y}$$

$$\frac{0 - 1}{2 - 0 - 0}$$

$$\frac{-1}{2} = -\frac{1}{2}$$

$\therefore$ Hence, the limit of the given function is $-\frac{1}{2}$.

Derivatives and Differentiation

Definition: The Given function $f(x)$ is said to be a differentiable function then we need to find

$$f'(x) \text{ i.e., } f'(x) = \frac{d}{dx}(f(x))$$

Basic Differentiation formulae:

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(a) = 0 \text{ (where } a \text{ is constant)}$$

$$\frac{d}{dx}(ax) = a$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$\frac{d}{dx}(a^x) = a^x \log a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{a \log x}$$

$$\frac{d}{dx} (e^{ax}) = a \cdot e^{ax}$$

$$\frac{d}{dx} (-1/x) = -1/x^2$$

$$\frac{d}{dx} (-1/x^2) = -2/x^3$$

$$\frac{d}{dx} (u+v) = \frac{d}{dx} (u) + \frac{d}{dx} (v)$$

$$\frac{d}{dx} (u-v) = \frac{d}{dx} (u) - \frac{d}{dx} (v)$$

$$\frac{d}{dx} (u \cdot v) = u \cdot \frac{d}{dx} (v) + v \cdot \frac{d}{dx} (u)$$

$$\frac{d}{dx} (u/v) = \frac{v \cdot \frac{d}{dx} (u) - u \cdot \frac{d}{dx} (v)}{v^2}$$

1) Find the derivative of the function

$$f(x) = 24x^3 + 12x^2 - 6$$

Given function $f(x) = 24x^3 + 12x^2 - 6$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx} (f(x))$$

$$= \frac{d}{dx} (24x^3 + 12x^2 - 6)$$

$$= \frac{d}{dx} (24x^3) + \frac{d}{dx} (12x^2) - \frac{d}{dx} (6)$$

$$= 24 \frac{d}{dx} (x^3) + 12 \frac{d}{dx} (x^2) - 0$$

$$= 24 \cdot 3 \cdot x^{3-1} + 12 \cdot 2 \cdot x^{2-1} - 0$$

$$= 72x^2 + 24x$$

2) find $f'(x)$, $f(x) = 2x^3 - 6x^2 - 18x + 21$

Given function $f(x) = 2x^3 - 6x^2 - 18x + 21$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx}(f(x))$$

$$= \frac{d}{dx}(2x^3 - 6x^2 - 18x + 21)$$

$$= \frac{d}{dx}(2x^3) - \frac{d}{dx}(6x^2) - \frac{d}{dx}(18x) + \frac{d}{dx}(21)$$

$$= 2 \cdot \frac{d}{dx}(x^3) - 6 \cdot \frac{d}{dx}(x^2) - 18 \cdot \frac{d}{dx}(x) + 0$$

$$= 2 \cdot 3x^{3-1} - 6 \cdot 2x^{2-1} - 18(1) + 0$$

$$= 6x^2 - 12x - 18$$

find the derivative of $x^6 - 6x^5 + 5/x^2 - \log x$

Given function $f(x) = x^6 - 6x^5 + 5/x^2 - \log x$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx}(x^6 - 6x^5 + 5/x^2 - \log x)$$

$$= \frac{d}{dx}(x^6) - \frac{d}{dx}(6x^5) + \frac{d}{dx}(5/x^2) - \frac{d}{dx}(\log x)$$

$$= \frac{d}{dx}(x^6) - 6 \frac{d}{dx}(x^5) + 5 \frac{d}{dx}(1/x^2) - \frac{d}{dx}(\log x)$$

$$= 6x^{6-1} - 6 \times 5 \cdot x^{5-1} + 5 \cdot (-2/x^3) - 1/x$$

$$= 6x^5 - 30x^4 - 10/x^3 - 1/x$$

$$4) \frac{d}{dx} \left(\frac{8}{x^8} - \frac{6}{x^6} + 2 \right)$$

Given function $f(x)$ is $\frac{d}{dx} \left(\frac{8}{x^8} - \frac{6}{x^6} + 2 \right)$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx} (f(x))$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{8}{x^8} - \frac{6}{x^6} + 2 \right)$$

$$= \frac{d}{dx} \left(\frac{8}{x^8} \right) - \frac{d}{dx} \left(\frac{6}{x^6} \right) + \frac{d}{dx} (2)$$

$$= 8 \cdot \frac{d}{dx} \left(\frac{1}{x^8} \right) - 6 \frac{d}{dx} \left(\frac{1}{x^6} \right) + 0$$

$$= 8 \cdot \left(-\frac{8}{x^9} \right) - 6 \left(-\frac{6}{x^7} \right)$$

$$= -\frac{64}{x^9} + \frac{36}{x^7}$$

$$5) \frac{d}{dx} (e^x + x^y \sec x)$$

Given function $f(x)$ is $\frac{d}{dx} (e^x + x^y \sec x)$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx} (f(x))$$

$$f'(x) = \frac{d}{dx} (e^x + x^y \sec x)$$

$$= \frac{d}{dx} (e^x) + \frac{d}{dx} (x^y \sec x)$$

$$= e^x + \left[x^y \cdot \frac{d}{dx} (\sec x) + \sec x \cdot \frac{d}{dx} (x^y) \right]$$

$$= e^x + [x^y (\sec x \cdot \tan x) + \sec x (y x^{y-1})]$$

$$= e^x + x^y \sec x \cdot \tan x + y x^{y-1} \sec x$$

6) $\frac{d}{dx}(x^3 \cdot \tan x)$

Given function $f(x)$ is $\frac{d}{dx}(x^3 \cdot \tan x)$

Derivative of $f(x)$ is $f'(x)$

$$f(x) = \frac{d}{dx}(x^3 \cdot \tan x)$$

$$= x^3 \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(x^3)$$

$$= x^3 (\sec^2 x) + \tan x (3x^2)$$

$$= x^3 \sec^2 x + 3x^2 \tan x$$

7) $\frac{d}{dx}(\sin 3x)$

Given function $f(x)$ is $\frac{d}{dx}(\sin 3x)$

Derivative of $f(x)$ is $f'(x)$

$$f(x) = \frac{d}{dx}(\sin 3x)$$

$$= \cos 3x \cdot \frac{d}{dx}(3x)$$

$$= 3 \cos 3x$$

8) $\frac{d}{dx}\left[\frac{3x+1}{7x+5}\right]$

Given function $f(x)$ is $\frac{d}{dx}\left[\frac{3x+1}{7x+5}\right]$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx}\left[\frac{3x+1}{7x+5}\right] = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{v^2}$$

$$= \frac{(7x+5) \cdot \frac{d}{dx}(3x+1) - (3x+1) \cdot \frac{d}{dx}(7x+5)}{(7x+5)^2}$$

$$= \frac{(7x+5) \cdot \left[\frac{d}{dx}(3x) + \frac{d}{dx}(1)\right] - (3x+1) \left[\frac{d}{dx}(7x) + \frac{d}{dx}(5)\right]}{(7x+5)^2}$$

$$= \frac{(7x+5) \cdot (3) - (3x+1)(7)}{(7x+5)^2}$$

$$= \frac{21x+15 - (21x+7)}{(7x+5)^7}$$

$$= \frac{21x+15 - 21x - 7}{(7x+5)^7}$$

$$= \frac{15-7}{(7x+5)^7}$$

$$= \frac{8}{(7x+5)^7}$$

9) $\frac{d}{dx} \left[\frac{x^4 - 2x + 3}{x^2} \right]$

Given function $f(x)$ is $\frac{x^4 - 2x + 3}{x^2}$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx} \left[\frac{x^4 - 2x + 3}{x^2} \right] = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{v^2}$$

$$u = x^4 - 2x + 3, \quad v = x^2$$

$$= \frac{x^2 \cdot \frac{d}{dx}(x^4 - 2x + 3) - (x^4 - 2x + 3) \cdot \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 \left[\frac{d}{dx}(x^4) - \frac{d}{dx}(2x) + \frac{d}{dx}(3) \right] - (x^4 - 2x + 3) \cdot \frac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^2(4x^3 - 2 + 0) - (x^4 - 2x + 3)(2x)}{x^4}$$

$$= \frac{4x^5 - 2x^2 - [2x^5 - 4x^2 + 6x]}{x^4}$$

$$= \frac{4x^5 - 2x^2 - 2x^5 + 4x^2 - 6x}{x^4}$$

$$= \frac{2x^5 + 2x^2 - 6x}{x^4}$$

10) find the derivate of $y = \sin(x^2 + 2x + 1)^3$

Given function $y = \sin(x^2 + 2x + 1)^3$

Derivative of $f(x)$ is $f'(x)$

$$f(x) = \frac{d}{dx} [\sin(x^2 + 2x + 1)^3]$$

$$= \cos(x^2 + 2x + 1)^3 \cdot \frac{d}{dx} (x^2 + 2x + 1)^3$$

$$= \cos(x^2 + 2x + 1)^3 \cdot 3(x^2 + 2x + 1)^2 \cdot \frac{d}{dx} (x^2 + 2x + 1)$$

$$= \cos(x^2 + 2x + 1)^3 \cdot 3(x^2 + 2x + 1)^2 \cdot (2x + 2)$$

11) $y = (ax + b)^n$

Given function $y = (ax + b)^n$

Derivative of $f(x)$ is $f'(x)$

$$f'(x) = \frac{d}{dx} (ax + b)^n$$

$$= n(ax + b)^{n-1} \cdot \frac{d}{dx} (ax + b)$$

$$= n(ax + b)^{n-1} \cdot a$$

$$= an(ax + b)^{n-1}$$

Maxima and Minima

Steps for finding Maxima & Minima

Step: 1 find $f'(x)$ to get the roots of given function.

Step: 2 find the roots for $f'(x) = 0$.

Step: 3 find $f''(x)$

Step: 4 substitute x values in $f''(x)$ & $f(x)$.

1) find maximum and minimum value of

$$f(x) = 2x^3 - 6x^2 - 18x + 21$$

Given function $f(x) = 2x^3 - 6x^2 - 18x + 21$

Step: 0 (Case: 1)

Step: 1 find $f'(x)$ to get the roots of given function. Substitute $x=3$ in $f''(x)$

$$f'(x) = 12x - 12$$

$$f''(3) = 12(3) - 12$$

$$= 36 - 12$$

$$= 24 > 0$$

$$f(x) = 2x^3 - 6x^2 - 18x + 21$$

$$f'(x) = \frac{d}{dx}(f(x))$$

$$= \frac{d}{dx}(2x^3 - 6x^2 - 18x + 21)$$

$$= \frac{d}{dx}(2x^3) - \frac{d}{dx}(6x^2) - \frac{d}{dx}(18x) + \frac{d}{dx}(21)$$

$$= 6x^2 - 12x - 18$$

$f'(x) > 0$ which is maximum value of $f(x)$ is 24, at $x=3$

for the given function $f(x)$

Sub $x=3$ in $f(x)$

$$f(x) = 2x^3 - 6x^2 - 18x + 21$$

$$f(3) = 2(3)^3 - 6(3)^2 - 18(3) + 21$$

$$= 2(27) - 6(9) - 54 + 21$$

$$= 54 - 54 - 54 + 21$$

$$= -33 < 0$$

$f(x) < 0$ which is minimum value at $x=3$ is -33.

Case: 2

Substitute $x=-1$ in $f'(x)$

$$f'(x) = 12x - 12$$

$$f'(-1) = 12(-1) - 12$$

$$= -12 - 12$$

$$= -24 < 0$$

$f'(x) < 0$ which is minimum value at $x=-1$ is -24.

Sub $x=-1$ in $f(x)$

$$f(x) = 2x^3 - 6x^2 - 18x + 21$$

$$f(-1) = 2(-1)^3 - 6(-1)^2 - 18(-1) + 21$$

$$= 2(-1) - 6(1) + 18 + 21$$

$$= -2 - 6 + 18 + 21$$

$$= 31 > 0$$

Max val of $f(x)$ is 31 at $x=-1$

Step: 2: find the roots of $f'(x)=0$

$$f'(x) = 0$$

$$6x^2 - 12x - 18 = 0$$

$$6(x^2 - 2x - 3) = 0$$

$$x^2 - 2x - 3 = 0$$

$$x^2 + 1x - 3x - 3 = 0$$

$$x(x+1) - 3(x+1) = 0$$

$$(x+3)(x+1) = 0$$

$$x+3=0 \quad | \quad x+1=0$$

$$x=-3 \quad | \quad x=-1$$

Step: 3 find $f''(x)$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$f''(x) = \frac{d}{dx}(6x^2 - 12x - 18)$$

$$= \frac{d}{dx}(6x^2) - \frac{d}{dx}(12x) - \frac{d}{dx}(18)$$

$$= 12x - 12$$

$$2) f(x) = 4x^3 - 18x^2 + 24x - 7$$

Given function $f(x) = 4x^3 - 18x^2 + 24x - 7$

Step: 1 find $f'(x)$ to get the roots of $f(x)$

$$f'(x) \text{ is } \frac{d}{dx}(f(x))$$

$$f'(x) = \frac{d}{dx}(4x^3 - 18x^2 + 24x - 7)$$

$$= \frac{d}{dx}(4x^3) - \frac{d}{dx}(18x^2) + \frac{d}{dx}(24x) - \frac{d}{dx}(7)$$

$$= 12x^2 - 36x + 24$$

Step: 2 find the roots of $f'(x) = 0$

$$12x^2 - 36x + 24 = 0$$

$$12(x^2 - 3x + 2) = 0$$

$$x^2 - 3x + 2 = 0$$

$$x^2 - x - 2x + 2 = 0$$

$$x(x-1) - 2(x-1) = 0$$

$$(x-1)(x-2) = 0$$

$$x-1=0 \quad | \quad x-2=0$$

$$x=1 \quad | \quad x=2$$

Step: 3 find $f''(x)$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$f''(x) = \frac{d}{dx}(12x^2 - 36x + 24)$$

$$= \frac{d}{dx}(12x^2) - \frac{d}{dx}(36x) + \frac{d}{dx}(24)$$

$$= 24x - 36$$

Step 4 (Case: 1)

Substitute $x=1$ in $f''(x)$

$$f''(1) = 24(1) - 36$$

$$f''(1) = 24(1) - 36$$

$$= 24 - 36$$

$$= -12 < 0$$

$f''(x) < 0$ which is minimum

Value at $x=1$ is -12

Substitute $x=1$ in $f(x)$

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f(1) = 4(1)^3 - 18(1)^2 + 24(1) - 7$$

$$= 4 - 18 + 24 - 7$$

$$= 28 - 25$$

$$= 3 > 0$$

$f(x) > 0$ which is maximum

Value at $x=1$ is 3.

Case: 2 Sub $x=2$ in $f''(x)$

$$f''(x) = 24x - 36$$

$$f''(2) = 24(2) - 36$$

$$= 48 - 36 = 12 > 0$$

$f''(x) > 0$ which is max value

at $x=2$ is 12

Sub $x=2$ in $f(x)$

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f(2) = 4(2)^3 - 18(2)^2 + 24(2) - 7$$

$$= 4(8) - 18(4) + 48 - 7$$

$$= 32 - 72 + 48 - 7$$

$$= 1 > 0$$

$f(x) > 0$ which is min value

at $x=2$ is 1

$$3) \frac{d}{dx} (4x^3 - 3x^2 - 18x + 12)$$

Given function $f(x)$ is $4x^3 - 3x^2 - 18x + 12$

Step: 1 find $f'(x)$ to get roots of $f(x)$

$$f'(x) = \frac{d}{dx} (f(x))$$

$$f'(x) = \frac{d}{dx} (4x^3 - 3x^2 - 18x + 12)$$

$$= \frac{d}{dx} (4x^3) - \frac{d}{dx} (3x^2) - \frac{d}{dx} (18x) + 12$$

$$= 12x^2 - 6x - 18$$

Step: 2 find the roots of $f'(x) = 0$

$$12x^2 - 6x - 18 = 0$$

$$6(2x^2 - x - 3) = 0$$

$$2x^2 - x - 3 = 0$$

$$2x^2 + 2x - 3x - 3 = 0$$

$$2x(x+1) - 3(x+1) = 0$$

$$(2x-3)(x+1) = 0$$

$$\begin{array}{l|l} 2x-3 & x+1=0 \\ 2x-3=0 & \\ 2x=3 & x=-1 \\ x=3/2 & \end{array}$$

Step: 3 find $f''(x)$

$$f''(x) = \frac{d}{dx} (f'(x))$$

$$= \frac{d}{dx} (12x^2 - 6x - 18)$$

$$= \frac{d}{dx} (12x^2) - \frac{d}{dx} (6x) - \frac{d}{dx} (18)$$

$$= 24x - 6$$

Step 4 (Case: 1)

Sub $x = 3/2$ in $f''(x)$

$$\begin{aligned}f''(x) &= 24x - 6 \\&= 24(3/2) - 6 \\&= 36 - 6 \\&= 30 > 0\end{aligned}$$

As $f''(x) > 0$, which is max value at $x = 3/2$ is 30

Sub $x = 3/2$ in $f(x)$

$$\begin{aligned}f(x) &= 4x^3 - 3x^2 - 18x + 12 \\f(3/2) &= 4(3/2)^3 - 3(3/2)^2 - 18(3/2) + 12 \\&= 4\left(\frac{27}{8}\right) - 3\left(\frac{9}{4}\right) - 18\left(\frac{3}{2}\right) + 12 \\&= \frac{27}{2} - \frac{27}{4} - 27 + 12 \\&= \frac{(27 \times 2) - 27 - (27 \times 4) + 12(4)}{4} \\&= \frac{54 - 27 - 108 + 48}{4} \\&= -\frac{33}{4} \\&= -8.2 < 0\end{aligned}$$

$f(x) < 0$ which is min value at $x = 3/2$ is -8.2

Case: 2 substitute $x = -1$ in $f''(x)$

$$\begin{aligned}f''(x) &= 24x - 6 \\&= 24(-1) - 6 \\&= -24 - 6 \\&= -30 < 0\end{aligned}$$

$f''(x) < 0$ which is min value at $x = -1$ is -30

Sub $x = -1$ in $f(x)$

$$\begin{aligned}f(x) &= 4x^3 - 3x^2 - 18x + 12 \\&= 4(-1)^3 - 3(-1)^2 - 18(-1) + 12 \\&= 4(-1) - 3(1) + 18 + 12 = 30 - 7 = 23 > 0\end{aligned}$$

$f(x) > 0$ which is max value at $x = -1$ is 23.

home work sums

1) $\frac{d}{dx} (2x^3 + 3x^2 - 12x + 3)$, $\frac{d}{dx} (2x^3 - 9x^2 + 12x + 5)$

Given function $f(x) = 2x^3 + 3x^2 - 12x + 3$

Step: 1 find $f'(x)$ to get the roots of $f(x)$

$f'(x)$ is $\frac{d}{dx} (f(x))$

$f'(x) = \frac{d}{dx} (2x^3 + 3x^2 - 12x + 3)$

$= \frac{d}{dx} (2x^3 + 3x^2 - 12x + 3)$

$= \frac{d}{dx} (2x^3) + \frac{d}{dx} (3x^2) - \frac{d}{dx} (12x) + \frac{d}{dx} (3)$

$= 6x^2 + 6x - 12$

Step: 2 find the roots of $f'(x) = 0$

$6x^2 + 6x - 12 = 0$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0$

$x = 1, -2$

Step: 3 find $f''(x)$

$f''(x)$ is $\frac{d}{dx} (f'(x))$

$f''(x) = \frac{d}{dx} (6x^2 + 6x - 12)$

$= \frac{d}{dx} (6x^2) + \frac{d}{dx} (6x) - \frac{d}{dx} (12)$

$= 12x + 6$

Step: 4 (Case: 1)

Sub $x=+1$ in $f'(x)$

$$f'(x) = 12x + 6$$

$$f'(1) = 12(1) + 6$$

$$= 12 + 6$$

$$= 18 > 0$$

$f'(x) > 0$, which is maximum value at $x=+1$ is 18.

Sub $x=+1$ in $f(x)$

$$f(x) = 2x^3 + 3x^2 - 12x + 3$$

$$= 2(1)^3 + 3(1)^2 - 12(1) + 3$$

$$= 2 + 3 - 12 + 3$$

$$= -4 < 0$$

$f(x) < 0$ which is minimum value at $x=+1$ is -4

Case: 2

Sub $x=-2$ in $f'(x)$

$$f'(x) = 12x + 6$$

$$f'(-2) = 12(-2) + 6$$

$$= -24 + 6$$

$$= -18 < 0$$

$f'(x) < 0$ which is minimum value at $x=-2$ is -18

Sub $x=-2$ in $f(x)$

$$f(x) = 2x^3 + 3x^2 - 12x + 3$$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 3$$

$$= 2(-8) + 3(4) + 24 + 3$$

$$= -16 + 12 + 24 + 3$$

$$= 23 > 0$$

$f(x) > 0$ which is max value at $x=-2$ is 23.

Q $\frac{d}{dx}(2x^3 - 9x^2 + 12x + 5)$

Given function $f(x)$ is $2x^3 - 9x^2 + 12x + 5$

Step:1 find $f'(x)$ to get the roots of $f(x)$

$$f'(x) = \frac{d}{dx}(f(x))$$

$$f'(x) = \frac{d}{dx}(2x^3 - 9x^2 + 12x + 5)$$

$$= \frac{d}{dx}(2x^3) - \frac{d}{dx}(9x^2) + \frac{d}{dx}(12x) + \frac{d}{dx}(5)$$

$$= 6x^2 - 18x + 12$$

Step:2 find the roots of $f'(x) = 0$

$$6x^2 - 18x + 12 = 0$$

$$6(x^2 - 3x + 2) = 0$$

$$x^2 - 3x + 2 = 0$$

$$x^2 - x - 2x + 2 = 0$$

$$x(x-1) - 2(x-1) = 0$$

$$(x-1)(x-2) = 0$$

$$x-1=0 \quad | \quad x-2=0$$

$$x=1 \quad | \quad x=2$$

Step:3 find $f''(x)$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$f''(x) = \frac{d}{dx}(6x^2 - 18x + 12)$$

$$= \frac{d}{dx}(6x^2) - \frac{d}{dx}(18x) + \frac{d}{dx}(12)$$

$$= 12x - 18$$

Qp: 4 (Case: 1)

Substitute $x=1$ in $f''(x)$

$$f''(x) = 12x - 18$$

$$f''(1) = 12(1) - 18$$

$$= 12 - 18$$

$$= -6 < 0$$

$f''(x) < 0$, which is minimum value at $x=1$ is -6

Substitute $x=1$ in $f(x)$

$$f(x) = 2x^3 - 9x^2 + 12x + 5$$

$$f(1) = 2(1)^3 - 9(1)^2 + 12(1) + 5$$

$$= 2 - 9 + 12 + 5$$

$$= 10 > 0$$

$f(x) > 0$, which is maximum value at $x=1$ is 10

Case: 2

Substitute $x=2$ in $f''(x)$

$$f''(x) = 12x - 18$$

$$f''(2) = 12(2) - 18$$

$$= 24 - 18$$

$$= 6 > 0$$

$f''(x) > 0$, which is minimum value at $x=2$ is 6

Sub $x=2$ in $f(x)$

$$f(x) = 2x^3 - 9x^2 + 12x + 5$$

$$f(2) = 2(2)^3 - 9(2)^2 + 12(2) + 5$$

$$= 2(8) - 9(4) + 24 + 5$$

$$= 16 - 36 + 24 + 5$$

$$= 9 > 0$$

$f(x) > 0$, which is max value at $x=2$ is 9

Integration formulae

$$\int dx = x$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \Rightarrow \text{eg: } \int x^2 dx = \frac{x^{2+1}}{2+1} = x^3/3$$

$$\int 1/x \cdot dx = \log x$$

$$\int \sin x \cdot dx = -\cos x$$

$$\int \cos x \cdot dx = \sin x$$

$$\int \tan x \cdot dx = \log(\sec x)$$

$$\int a^x \cdot dx = \frac{a^x}{\log a}$$

$$\int \cot x \cdot dx = \log(\tan x)$$

$$\int \sec x \cdot dx = \log(\sec x + \tan x)$$

$$\int \csc x \cdot dx = \log(\csc x - \cot x)$$

$$\int \sec^2 x \cdot dx = \tan x$$

$$\int \csc^2 x \cdot dx = -\cot x$$

$$\int e^x \cdot dx = e^x$$

$$\int e^{ax} \cdot dx = \frac{e^{ax}}{a}$$

$$\int \sin ax \cdot dx = \frac{-\cos ax}{a}$$

$$\int \sec x \cdot \tan x \cdot dx = \sec x$$

$$\int \csc x \cdot \cot x \cdot dx = -\csc x$$

$$\int uv \cdot dx = u \int v \cdot dx - \int \left[\frac{d}{dx} u \right] \cdot (v \cdot dx) \cdot dx + C$$

Trigonometric Identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\operatorname{cosec}^2 \theta - \sec^2 \theta = 1$$

$$\frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\frac{\cos \theta}{\sin \theta} = \cot \theta$$

$$\frac{1}{\cos \theta} = \sec \theta$$

$$\frac{1}{\sin \theta} = \operatorname{cosec} \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

$$e^{\log f(x)} = f(x)$$

1) Evaluate $\int (x^3 + 3x^2 + 2) \cdot dx$

Given Integration $\int (x^3 + 3x^2 + 2) \cdot dx$

$$= \int x^3 \cdot dx + \int 3x^2 \cdot dx + \int 2 \cdot dx$$

$$= \int x^3 \cdot dx + \int 3x^2 \cdot dx + 2 \int 1 \cdot dx$$

$$= \frac{x^{3+1}}{3+1} + \frac{3x^{2+1}}{2+1} + 2x$$

$$= \frac{x^4}{4} + \frac{3x^3}{3} + 2x$$

2) $\int (3x^2 + 2x + 5) \cdot dx$

$$\int 3x^2 \cdot dx + \int 2x \cdot dx + \int 5 \cdot dx$$

$$3 \int x^2 \cdot dx + 2 \int x \cdot dx + 5 \int 1 \cdot dx$$

$$3 \cdot \frac{x^{2+1}}{2+1} + 2 \cdot \frac{x^{1+1}}{1+1} + 5 \cdot x$$

$$= 3\frac{x^3}{3} + 2\frac{x^4}{2} + 5x$$

$$= x^3 + x^4 + 5x.$$

$$3) \int (\sec^2 x - e^x + \sin x) \cdot dx$$

$$\text{Given Integration } \int (\sec^2 x - e^x + \sin x) dx$$

$$\int \sec^2 x dx - \int e^x dx + \int \sin x \cdot dx$$

$$\tan x - e^x - \cos x.$$

$$4) \int (x^2 + 2x + 3) \cdot dx$$

$$\text{Given Integration } \int (x^2 + 2x + 3) \cdot dx$$

$$\int x^2 dx + \int 2x dx + \int 3 \cdot dx$$

$$\int x^2 dx + 2 \int x dx + 3 \int 1 dx$$

$$\frac{x^{2+1}}{2+1} + 2 \cdot \frac{x^{1+1}}{1+1} + 3 \cdot x$$

$$\frac{x^3}{3} + 2\frac{x^2}{2} + 3x$$

$$\frac{x^3}{3} + x^2 + 3x$$

$$5) \int (2x^3 + 3/x^2 - 1/x) \cdot dx$$

$$\text{Given Integration } \int (2x^3 + 3/x^2 - 1/x) \cdot dx$$

$$\int 2x^3 dx + \int 3/x^2 dx - \int 1/x dx$$

$$2 \int x^3 dx + 3 \int 1/x^2 dx - \log x$$

$$2 \left(\frac{x^{3+1}}{3+1} \right) + 3 \int x^{-2} - \log x$$

$$2 \left(\frac{x^4}{4} \right) + 3 \left(\frac{x^{-2+1}}{-2+1} \right) - \log x$$

$$\frac{x^4}{2} + 3 \left(\frac{x^{-1}}{-1} \right) - \log x$$

$$\frac{x^4}{2} - 3/x - \log x.$$

Additional Problems:

1) $\int (1-x)(4-3x)(3+2x) \cdot dx$

Given Integration $\int (1-x)(4-3x)(3+2x) \cdot dx$

$$\int (1-x)(4-3x)(3+2x) \cdot dx$$

$$\int ((1 \cdot 4) - (1 \cdot 3x) - 4x + x \cdot 3x)(3+2x) \cdot dx$$

$$\int (4-3x-4x+3x^2)(3+2x) \cdot dx$$

$$\int (4-7x+3x^2)(3+2x) \cdot dx$$

$$\int (3(4-7x+3x^2) + 2x(4-7x+3x^2)) \cdot dx$$

$$\int (12-21x+9x^2+8x-14x^2+6x^3) \cdot dx$$

$$\int (6x^3-5x^2-13x+12) \cdot dx$$

$$\int 6x^3 \cdot dx - \int 5x^2 \cdot dx - \int 13x \cdot dx + \int 12 \cdot dx$$

$$6 \int x^3 \cdot dx - 5 \int x^2 \cdot dx - 13 \int x \cdot dx + 12 \int 1 \cdot dx$$

$$6 \cdot \frac{x^{3+1}}{3+1} - 5 \cdot \frac{x^{2+1}}{2+1} - 13 \cdot \frac{x^{1+1}}{1+1} + 12x$$

$$6 \frac{x^4}{4} - 5 \frac{x^3}{3} - 13 \frac{x^2}{2} + 12x$$

$$\frac{3x^4}{2} - \frac{5x^3}{3} - \frac{13x^2}{2} + 12x$$

2) $\int \frac{\log(1+\tan^2 x)}{e^x} \cdot dx$

Given Integration $\int e^{\log(1+\tan^2 x)} \cdot dx$

$$\int e^{\log(1+\tan^2 x)} \cdot dx = \int (1+\tan^2 x)$$

$$\downarrow$$

$$e^{\log(x)} = f(x) = \sec^2 x$$

$$= \tan x$$

$$3) \int \frac{\sin^2 x}{1 + \cos 2x}$$

* Given Integration $\int \frac{\sin^2 x}{1 + \cos 2x}$

$$= \int \frac{\sin^2 x}{1 + 2\cos^2 x - 1}$$

$$= \int \frac{\sin^2 x}{2\cos^2 x}$$

$$= \frac{1}{2} \int \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{2} \int \tan^2 x$$

$$= \frac{1}{2} \int (\sec^2 x - 1) dx$$

$$= \frac{1}{2} \int \sec^2 x - \frac{1}{2} \int 1 \cdot dx$$

$$= \frac{1}{2} \tan x - \frac{1}{2} x$$

$$4) \int \sec^2 x + \operatorname{cosec}^2 x \cdot dx$$

Given Integration $\int \sec^2 x + \operatorname{cosec}^2 x \cdot dx$

$$\int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) \cdot dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \cdot dx$$

$$\int \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{1}{\sin^2 x \cos^2 x}$$

$$= \int \frac{1}{\cos^2 x} + \int \frac{1}{\sin^2 x}$$

$$= \int \sec^2 x + \int \operatorname{cosec}^2 x$$

$$= \tan x - \cot x + C$$

$$\int \frac{1}{1-\sin x} \cdot dx$$

Given Integration $\int \frac{1}{1-\sin x} \cdot dx$

Rationalise the given Integration

$$\int \left(\frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} \right) \cdot dx$$

Denominator is in the form of $(a-b)(a+b)$ is a^2-b^2

$$\int \left(\frac{1(1+\sin x)}{1^2 - \sin^2 x} \right) \cdot dx$$

$$\int \left(\frac{1+\sin x}{1-\sin^2 x} \right) \cdot dx$$

$$= \int \frac{1+\sin x}{\cos^2 x} \cdot dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) \cdot dx$$

$$\left(= \int \frac{1}{\cos^2 x} \cdot dx + \int \frac{\sin x}{\cos^2 x} \cdot dx \right) = \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x \cdot \cos x} \right) \cdot dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \tan x \cdot \frac{1}{\cos x} \right) \cdot dx$$

$$= \int \frac{1}{\cos^2 x} \cdot dx + \int \tan x \cdot \sec x \cdot dx$$

$$= \int \sec^2 x \cdot dx + \int \tan x \cdot \sec x \cdot dx$$

$$= \tan x + \sec x$$

$$\int \frac{1}{1-\cos x} \cdot dx$$

Given Integration $\int \frac{1}{1-\cos x} \cdot dx$

Rationalise the given integration

$$= \int \left(\frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} \right) \cdot dx$$

$$= \int \frac{1(1+\cos x)}{(1-\cos x)(1+\cos x)} \cdot dx$$

Denominator is in the form of $(a-b)(a+b) = a^2 - b^2$

$$\int \left(\frac{1+\cos x}{1^2 - \cos^2 x} \right) \cdot dx$$

$$\int \left(\frac{1+\cos x}{\sin^2 x} \right) \cdot dx$$

$$\int \frac{1}{\sin^2 x} \cdot dx + \int \frac{\cos x}{\sin^2 x} \cdot dx$$

$$\int \operatorname{cosec}^2 x \cdot dx + \int \frac{\cos x}{\sin x \cdot \sin x} \cdot dx$$

$$= -\cot x + \int \cot x \cdot \operatorname{cosec} x \cdot dx$$

$$= -\cot x - \operatorname{cosec} x$$

$$+ \int \left(\frac{x^2 + 2x + 3}{x^4} \right) \cdot dx$$

Given Integration $\int \left(\frac{x^2 + 2x + 3}{x^4} \right) \cdot dx$

$$= \int \frac{x^2}{x^4} \cdot dx + \int \frac{2x}{x^4} \cdot dx + \int \frac{3}{x^4} \cdot dx$$

$$\int \frac{1}{x^2} \cdot dx + \int \frac{2}{x^3} \cdot dx + \int \frac{3}{x^4} \cdot dx$$

$$\int x^{-2} \cdot dx + \int 2x^{-3} \cdot dx + \int 3x^{-4} \cdot dx$$

$$\int \frac{x^{-2+1}}{-2+1} + 2 \int \frac{x^{-3+1}}{-3+1} + 3 \int \frac{x^{-4+1}}{-4+1} \cdot dx$$

$$\frac{x^{-1}}{-1} + 2 \cdot \frac{x^{-2}}{-2} + 3 \cdot \frac{x^{-3}}{-3}$$

$$= -x^{-1} + \frac{x^{-2}}{-2} + \frac{x^{-3}}{-3}$$

$$= -\frac{1}{x} - \frac{1}{2}x^{-2} - \frac{1}{3}x^{-3}$$

$$* \int \left(\frac{1-\sqrt{x}}{x} \right) \cdot dx$$

$$\text{Given Integration } \int \left(\frac{1-\sqrt{x}}{x} \right) \cdot dx$$

$$\int \left(\frac{1-\sqrt{x}}{x} \right) dx$$

$$\int \left(\frac{1}{x} - \frac{\sqrt{x}}{x} \right) \cdot dx$$

$$\int \frac{1}{x} \cdot dx - \int \frac{1}{\sqrt{x}} \cdot dx$$

$$= \log x - \int \frac{1}{x^{1/2}} \cdot dx$$

$$= \log x - \int x^{-1/2} \cdot dx$$

$$= \log x - \frac{x^{-1/2+1}}{-1/2+1}$$

$$= \log x - \frac{x^{1/2}}{1/2}$$

$$= \log x - \frac{\sqrt{x}}{1/2}$$

$$= \log x - 2\sqrt{x}$$

$$* \int (x+\sqrt{x})^3 \cdot dx$$

$$\text{Given Integration } \int (x+\sqrt{x})^3 \cdot dx$$

It is in the form of $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\int (x+\sqrt{x})^3 = \int (x^3 + 3(x^2)(\sqrt{x}) + 3 \cdot x(\sqrt{x})^2 + (\sqrt{x})^3) \cdot dx$$

$$= \int (x^3 + 3x + 3\sqrt{x} + \sqrt{x}^3) \cdot dx$$

$$= \int x^3 \cdot dx + \int 3x \cdot dx + \int 3\sqrt{x} \cdot dx + \int \sqrt{x}^3 \cdot dx$$

$$\int x^3 \cdot dx + 3 \int x \cdot dx + 3 \int \frac{1}{x} \cdot dx + \int \frac{1}{x^3} \cdot dx.$$

$$\frac{x^{3+1}}{3+1} + 3 \cdot \frac{x^{1+1}}{1+1} + 3 \log x + \frac{x^{-3+1}}{-3+1}$$

$$\frac{x^4}{4} + \frac{3}{2} x^2 + 3 \log - \frac{1}{2x^2}.$$

$$+ \int \sqrt{1+\sin x} \cdot dx$$

$$* \int \text{Given Integration } \int \sqrt{1+\sin 2x} \cdot dx.$$

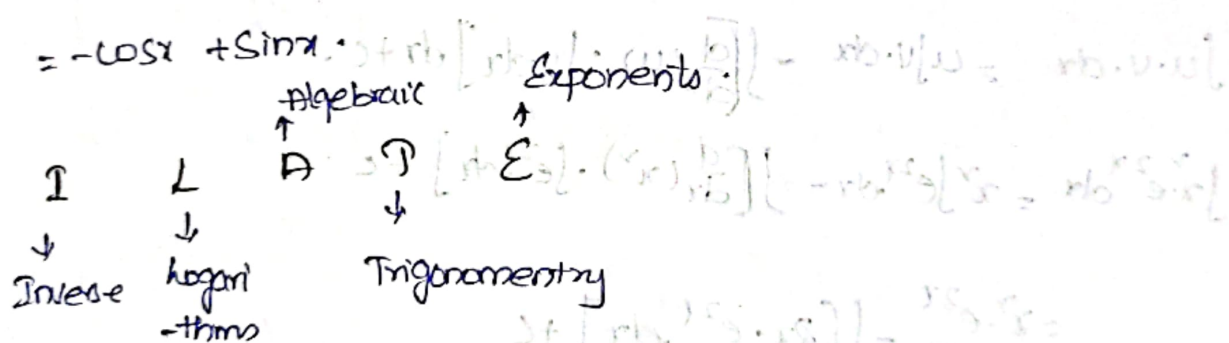
$$= \int (\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cdot \cos x}) \cdot dx.$$

$$= \int (\sqrt{(\sin x + \cos x)^2}) \cdot dx$$

$$= \int (\sin x + \cos x) \cdot dx.$$

$$= \int \sin x \cdot dx + \int \cos x \cdot dx.$$

$$= -\cos x + \sin x.$$



$$\int u \cdot v \cdot dx = u \cdot \int v \cdot dx - \int \left(\frac{d}{dx} (u) \cdot \int v \cdot dx \right) \cdot dx + C$$

$$* \int x \cdot \cos x.$$

$$u=x, v=\cos x.$$

$$\int u \cdot v = u \cdot \int v \cdot dx - \int \left(\frac{d}{dx} (u) \cdot \int v \cdot dx \right) \cdot dx + C$$

$$= x \cdot \int \cos x \cdot dx - \int \left(\frac{d}{dx} (x) \cdot \int \cos x \cdot dx \right) \cdot dx + C$$

$$= -x \cdot \sin x - \int 1 (\sin x) \cdot dx + C$$

$$= -x \sin x - (-\cos x) + C$$

$$= -x \sin x + \cos x + C.$$

$$* \int x \cdot e^x$$

Given Integration $\int x \cdot e^x \cdot dx$

$$u = x, \quad v = e^x$$

$$\int u \cdot v \cdot dx = u \cdot v \cdot dx - \int \left[\frac{d}{dx}(u) \cdot \int v \cdot dx \right] dx + C$$

$$= x \cdot e^x \cdot dx - \int \left[\frac{d}{dx}(x) \cdot \int e^x \cdot dx \right] \cdot dx + C$$

$$= x \cdot e^x - \int 1 \cdot e^x \cdot dx + C$$

$$= x \cdot e^x - e^x \cdot dx + C$$

$$= e^x(x-1) + C$$

$$* \int x^2 \cdot e^{2x} \cdot dx$$

Given Integration $\int x^2 \cdot e^{2x}$

$$u = x^2, \quad v = e^{2x}$$

$$\int u \cdot v \cdot dx = u \cdot v \cdot dx - \int \left[\frac{d}{dx}(u) \cdot \int v \cdot dx \right] dx + C$$

$$\int x^2 \cdot e^{2x} \cdot dx = x^2 \cdot e^{2x} - \int \left[\frac{d}{dx}(x^2) \cdot \int e^{2x} \cdot dx \right] + C$$

$$= x^2 \cdot \frac{e^{2x}}{2} - \int \left[2x \cdot \frac{e^{2x}}{2} \cdot dx \right] + C$$

$$= \frac{x^2 \cdot e^{2x}}{2} - \int x \cdot e^{2x} \cdot dx + C$$

$$= \frac{x^2 \cdot e^{2x}}{2} - \left[x \cdot \frac{e^{2x}}{2} - \int \left[\frac{d}{dx}(x) \cdot \int e^{2x} \cdot dx \right] \right] + C$$

$$= \frac{x^2 \cdot e^{2x}}{2} - \left[x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot dx \right] + C$$

$$= \frac{x^2 \cdot e^{2x}}{2} - \left[\frac{x \cdot e^{2x}}{2} - \frac{e^{2x}}{4} \right] + C$$

$$= \frac{x^2 \cdot e^{2x}}{2} - \frac{x \cdot e^{2x}}{2} + \frac{e^{2x}}{4} + C$$

$$\frac{x \cdot e^{2x}}{2} (x-1) + \frac{e^{2x}}{4} + C.$$

$$* \int x^y \cdot \sin 2x \, dx.$$

$$\text{Given Integration } \int x^y \cdot \sin 2x \, dx.$$

$$u = x^y, \quad v = \sin 2x$$

$$\int u \cdot v \, dx = u \int v \, dx - \int \left[\frac{d}{dx}(u) \cdot \int v \, dx \right] + C$$

$$= x^y \cdot \int \sin 2x \, dx - \int \left[\frac{d}{dx}(x^y) \cdot \int \sin 2x \, dx \right] + C.$$

$$= x^y \left(-\frac{\cos 2x}{2} \right) - \int dx \cdot \left(-\frac{\cos 2x}{2} \right) + C$$

$$= -\frac{x^y \cos 2x}{2} - \int x \cdot \frac{\cos 2x}{2} + C.$$

$$= -\frac{x^y \cos 2x}{2} + \int x \cdot \cos 2x \, dx + C.$$

$$= -\frac{x^y \cos 2x}{2} + \left(x \int \cos 2x - \int \left[\frac{d}{dx}(x) \cdot \int \cos 2x \right] + C \right)$$

$$= \frac{-x^y \cos 2x}{2} + \left(x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \right) + C.$$

$$= \frac{-x^y \cos 2x}{2} + \left(\frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \right) + C.$$

$$= \frac{-x^y \cos 2x}{2} + \frac{x \sin 2x}{2} - \frac{1}{2} \left(-\frac{\cos 2x}{2} \right) + C.$$

$$= \frac{-x^y \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C.$$

$$* \int x^y \cdot \cos 2x \cdot dx.$$

$$\text{Given Integration } \int x^y \cdot \cos 2x \cdot dx.$$

$$u = x^y, \quad v = \cos 2x.$$

$$\int u v dx = u \int v dx - \int \left[\frac{d}{dx}(u) \cdot \int v dx \right] + C.$$

$$= x^y \int \cos 2x \cdot dx - \int \left[\frac{d}{dx}(x^y) \cdot \int \cos 2x \cdot dx \right] + C$$

$$= x^y \frac{\sin 2x}{2} - \int \left[2x \cdot \frac{\sin 2x}{2} \right] \cdot dx + C$$

$$= \frac{x^y \sin 2x}{2} - \int x \sin 2x \cdot dx + C.$$

$$= \frac{x^y \sin 2x}{2} - \left[x \int \sin 2x - \int \left[\frac{d}{dx}(x) \cdot \int \sin 2x \cdot dx \right] + C \right]$$

$$= \frac{x^y \sin 2x}{2} - \left[x \left(-\frac{\cos 2x}{2} \right) - \int 1 \cdot \frac{\cos 2x}{2} \cdot dx + C \right]$$

$$= \frac{x^y \sin 2x}{2} - \left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \right] + C.$$

$$= \frac{x^y \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} + C.$$

UNIT-3

PROGRESSIONS

Progressions broadly classified into 3 types

- * Arithmetic progression [A.P]
- * Geometric progression [G.P]
- * Harmonic Progression [H.P]

Arithmetic Progression

In General form, Arithmetic progression can be written as $a, a+d, a+2d, \dots, a+nd$.

where 'a' is the first term
'd' is the common difference.

* The n^{th} term of Arithmetic progression can be represented as

$$a_n = a + (n-1)d$$

The sum of the first 'n' terms of an A.P is $S_n = \frac{n}{2} [2a + (n-1)d]$

↳ For Arithmetic progression is $\frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, \dots$ find the Common difference and 7^{th} term of Arithmetic progression and 20^{th} term of Arithmetic progression.

Given series of Arithmetic progression is $\frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, \dots$

from the given problem

$$a = \frac{1}{4}$$

To find common difference

$$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3$$

$$d = t_2 - t_1$$

from the given problem

$$t_1 = 1/4, t_2 = -1/4$$

$$d = t_2 - t_1$$

$$= -1/4 - 1/4$$

$$= \frac{-1-1}{4}$$

$$= -2/4 = -1/2$$

$$d = -1/2$$

$$t_n = a + (n-1)d$$

To find the n^{th} term we use the formula of n^{th} term of Arithmetic progression

$$a_n = a + (n-1)d$$

$$[a_7 = a + (7-1)d]$$

$$= a + 6d$$

$$= 1/4 + 6(-1/2)$$

$$= 1/4 - 3$$

$$= \frac{1-12}{4}$$

$$= -11/4$$

To find the 20th term of A.P.

$$a_{20} = a + (n-1)d$$

$$= a + (20-1)d$$

$$= a + 19d$$

$$= 1/4 + 19(-1/2)$$

$$= 1/4 - 19/2$$

$$= \frac{1-38}{4}$$

$$= -\frac{37}{4}$$

∴ The 7th term of Arithmetic progression is $-\frac{11}{4}$
The 20th term of Arithmetic Progression is $-\frac{37}{4}$

Find 10th term and 14th term of the Arithmetic Progression 5, 1, -3, -7

Given Arithmetic progression 5, 1, -3, -7

First term from the given series

$$a = 5$$

To find common difference, $d = t_2 - t_1$

$$t_2 = 1, t_1 = 5$$

$$d = t_2 - t_1$$

$$= 1 - 5$$

$$= -4$$

The 10th term of Arithmetic Progression is

$$a_n = a + (n-1)d$$

$$a_{10} = a + (10-1)d$$

$$= a + 9d$$

$$= 5 + 9(-4)$$

$$= 5 - 36$$

$$= -31$$

The 14th term of Arithmetic Progression is

$$a_n = a + (n-1)d$$

$$a_{14} = a + (14-1)d$$

$$= a + 13d$$

$$= 5 + 13(-4)$$

$$= 5 - 52$$

$$= -47$$

Which term of the Arithmetic progression

3, 8, 13, 18, ... is 78

Given series of an Arithmetic progression is

3, 8, 13, 18, ...

from the given series.

$$a = 3, t_1 = 3, t_2 = 8$$

$$d (\text{common difference}) = t_2 - t_1$$

$$d = t_2 - t_1$$

$$= 8 - 3$$

$$d = 5$$

and also given $a_n = 78$

The n th term of an Arithmetic progression is

$$a_n = a + (n-1)d$$

$$78 = 3 + (n-1)5$$

$$78 - 3 = (n-1)5$$

$$75 = (n-1)5$$

$$\frac{75}{5} = (n-1)$$

$$15 = n-1$$

$$15+1 = n$$

$$n = 16$$

$\therefore$ 16th term of an Arithmetic progression is 16

Determine the Arithmetic progression whose

3rd term is 5 and 7th term is 9

Given Information

3rd term is 5

7th term is 9

3rd term is 5 $\Rightarrow a_3 = 5$ 7th term is 9 $\Rightarrow a_7 = 9$

nth term is $a_n = a + (n-1)d$

$$a_n = a + (n-1)d$$

$$a_7 = a + (7-1)d$$

$$a_3 = a + (3-1)d$$

$$9 = a + 6d \quad \text{--- eq ②}$$

$$a_3 = a + 2d$$

$$5 = a + 2d \quad \text{--- eq ①}$$

Solve Equation 1 & 2.

$$\begin{array}{rcl} a+2d & = & 5 \\ a+6d & = & 9 \\ \hline \end{array}$$

$$+4d = +4$$

$$4d = 4$$

$$d = 4/4$$

$$d = 1$$

Substitute $d=1$ in any one of the Equations

$$\text{Eq ①} \Rightarrow a + 2d = 5$$

$$a + 2(1) = 5$$

$$a + 2 = 5$$

$$a = 5 - 2$$

$$a = 3$$

We know that,

The general series of Arithmetic progression,

is $a, a+d, a+2d, a+3d, a+4d, \dots$

$$3, 3+1, 3+2(1), 3+3(1), 3+4(1), \dots$$

$$3, 4, 3+2, 3+3, 3+4, \dots$$

$$3, 4, 5, 6, \dots$$

$\therefore$ The series of Arithmetic progression is

$$3, 4, 5, 6, \dots$$

Find the 1st four terms of an A.P where first term is 10, common difference is 10 and also find the 14th term.

Given Information.

first term $a=10$

Common difference $d=10$

We know that

The general series of an A.P is

$$a, a+d, a+2d, a+3d, \dots$$

$$10, 10+10, 10+2(10), 10+3(10), \dots$$

$$10, 20, 30, 40, \dots$$

The 14th term of an A.P is

$$a_n = a + (n-1)d$$

$$a_{14} = a + (14-1)d$$

$$= a + 13d$$

$$= 10 + 13(10)$$

$$= 10 + 130$$

$$= 140$$

$\therefore$ The 14th term of an A.P is 140.

Find the 11th term of Arithmetic progression

$$-3, -\frac{1}{2}, 2, \dots$$

Given series of an A.P $-3, -\frac{1}{2}, 2, \dots$

$$a = -3, t_1 = -3, t_2 = -\frac{1}{2}$$

$$d = t_2 - t_1 = -\frac{1}{2} - (-3) = -\frac{1}{2} + 3 = \frac{-1+6}{2} = \frac{5}{2}$$

$$11\text{th term } a_n = a + (n-1)d$$

$$a_{11} = a + (11-1)d$$

$$= a + 10d$$

$$= -3 + 10\left(\frac{5}{2}\right) = -3 + 25 = 22$$

For what value of n are, the n^{th} terms of two A.P's $63, 65, 67, \dots$ & $3, 10, 17, \dots$ are

Given series of A.P's are.

$$63, 65, 67, \dots$$

$$3, 10, 17, \dots$$

from the given series

from the given series

$$a=63, t_1=63, t_2=65$$

$$a=3, t_2=10, t_1=3$$

$$d=t_2-t_1$$

$$d=t_2-t_1$$

$$=65-63$$

$$=10-3$$

$$=2$$

$$=7$$

n^{th} term of A.P is

n^{th} term of A.P is

$$a_n = a + (n-1)d$$

$$a_n = a + (n-1)d$$

$$= 63 + (n-1)2$$

$$= 3 + (n-1)7$$

$$= 63 + 2n - 2$$

$$= 3 + 7n - 7$$

$$= 2n + 61$$

$$= 7n - 4$$

In the problem statement they given that the n^{th} term of two arithmetic progression are equal.

$$\text{i.e., } 2n + 61 = 7n - 4$$

$$61 + 4 = 7n - 2n$$

$$65 = 5n$$

$$\frac{65}{5} = n$$

$$n = 13$$

Hence, the given two arithmetic progression are equal at 13^{th} term

The sum of first 14 terms of A.P is 1050 & its first term is 10 find the 20th term.

Given

sum of first 14 terms of A.P is 1050

$$S_n = 1050$$

and also given first term,

$$a = 10$$

We know that,

sum of Arithmetic Progression

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$S_{14} = \frac{14}{2} [2(10) + (14-1)d]$$

$$1050 = 7 [20 + 13d]$$

$$\frac{1050}{7} = 20 + 13d$$

$$150 = 20 + 13d$$

$$150 - 20 = 13d$$

$$130 = 13d$$

$$d = \frac{130}{13}$$

$$d = 10$$

20th term of an A.P is

$$a_n = a + (n-1)d$$

$$a_{20} = a + (20-1)d$$

$$= 10 + 19(10)$$

$$= 10 + 190$$

$$= 200$$

$\therefore$ The 20th term of an A.P is 200.

The sum of 4th term and 8th term is 24 &
sum of 6th term and 10th term is 44. then
find the first 3 terms.

Sol Given,
Sum of 4th term and 8th term is 24

4th term of an A.P.,
 $a_n = a + (n-1)d$

$$a_4 = a + (4-1)d \\ = a + 3d.$$

8th term of an A.P.,

$$a_n = a + (n-1)d$$

$$a_8 = a + (8-1)d \\ = a + 7d.$$

∴ Sum of 4th term and 8th term is 24

$$a + 3d + a + 7d = 24$$

$$2a + 10d = 24$$

$$2(a + 5d) = 24$$

$$a + 5d = 12. \quad \text{--- Eq ①}$$

And also given,

sum of 6th term and 10th term is 44

6th term of an A.P.

$$a_n = a + (n-1)d$$

$$a_6 = a + (6-1)d$$

$$= a + 5d$$

10th term of an A.P.

$$a_n = a + (n-1)d$$

$$a_{10} = a + (10-1)d$$

$$= a + 9d.$$

$\therefore$ sum of 6th term & 10th term is 44

$$a+5d + a+9d = 44$$

$$2a+14d = 44$$

$$2(a+7d) = 44$$

$$a+7d = \frac{44}{2}$$

$$a+7d = 22$$

- Eq - ②

Solve Eq ① & ②

$$a+5d = 12$$

$$a+7d = 22$$

$$+2d = +10$$

$$2d = 10$$

$$d = 10/2$$

$$d = 5$$

Substitute $d=5$ in any equations.

$$1^{st} \text{ Eq} \Rightarrow a+5d = 12$$

$$a+5(5) = 12$$

$$a+25 = 12$$

$$a = 12 - 25$$

$$a = -13$$

We know that

first three terms of an A.P. is $a, a+d, a+2d$

$$\text{first term } (a) = -13$$

$$\text{Second term } (a+d) = -13+5 = -8$$

$$\text{Third term } (a+2d) = -13+5(2) = -13+10 = -3.$$

$\therefore$ The three terms are $-13, -8, -3$.

The first term of an Arithmetic progression is 5 and 10th term is 29, then find the sum of 16 terms of Arithmetic progression.

Given,

first term of an A.P. (a) = 5

10th term of an A.P. is 29 i.e. $a_{10} = 29$

$$a_n = a + (n-1)d$$

$$a_{10} = a + (10-1)d$$

$$29 = a + 9d$$

$$29 = 5 + 9(d)$$

$$29 - 5 = 9d$$

$$24 = 9d$$

$$\frac{24}{9} = d$$

$$d = \frac{8}{3}$$

Sum of 16 terms of an A.P. is

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$S_{16} = \frac{16}{2} [2(5) + (16-1)\frac{8}{3}]$$

$$= 8 [10 + 15(\frac{8}{3})]$$

$$= 8 [10 + 40]$$

$$= 8(50)$$

$$= 400$$

$\therefore$ The sum of 16 terms of an A.P. is 400.

Geometric Progression.

The General form of Geometric Progression can be written as $a, ar, ar^2, \dots, ar^{n-1}$

The n th term of G.P is

$$a_n = a \cdot r^{n-1}$$

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} \dots$$

$r =$ Common Ratio.

The sum of n terms of G.P is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

1) Find the 20th term and n th term of the G.P

$$5/2, 5/4, 5/8, \dots$$

Given series of Geometric Progression.

$$5/2, 5/4, 5/8$$

first term $a = 5/2$

Common ratio $r = \frac{a_2}{a_1} = \frac{5/4}{5/2} = \frac{1}{2}$

$$r = \frac{1}{2}$$

20th term of Geometric Progression

$$a_n = a \cdot r^{n-1}$$

$$a_{20} = (5/2) \cdot (1/2)^{20-1}$$

$$= 5/2 \cdot (1/2)^{19}$$

$$= \frac{5}{2^{20}}$$

n th term of Geometric Progression

$$a_n = a \cdot r^{n-1}$$

$$= (5/2) \cdot (1/2)^{n-1}$$

$$= \frac{5}{2} \left(\frac{1}{2^{n-1}} \right)$$

When bases are equal powers can be added.

$$= 5 \left(\frac{1}{2 \cdot 2^{n-1}} \right)$$

$$= 5 \left(\frac{1}{2^{1+n-1}} \right)$$

$$= \frac{5}{2^n}$$

$\therefore$ the n^{th} term of Geometric Progression is $\frac{5}{2^n}$.

Which term of G.P is 128, in the Given Geometric Progression $2, 2\sqrt{2}, 4, \dots$

Given Geometric Progression. $2, 2\sqrt{2}, 4, \dots$

first term $a=2$.

$$\text{Common ratio}(r) = \frac{a_2}{a_1}$$

$$r = \frac{2\sqrt{2}}{2}$$

$$r = \sqrt{2}$$

n^{th} term of Geometric Progression.

$$a_n = a \cdot r^{n-1}$$

and also given that n^{th} term is 128

$$\text{i.e., } a_n = 128$$

$$a_n = a \cdot r^{n-1}$$

$$128 = a \cdot (r)^{n-1}$$

$$128 = 2(r)^{n-1}$$

$$\frac{128}{2} = (r)^{n-1}$$

$$64 = (r)^{n-1}$$

$$(r)^{12} = (r)^{n-1}$$

When bases are equal, powers are also equal.

$$12 = n-1$$

$$12+1 = n$$

$$n = 13$$

∴ The 13th term of Geometric Progression is 128

In a geometric Progression the 3rd term is 24 and 6th term is 192, find the 10th term.

Given, the 3rd term is 24 i.e., $a_3 = 24$

$$a_n = a \cdot r^{n-1}$$

$$a_3 = a \cdot r^{3-1}$$

$$a_3 = a \cdot r^2$$

$$24 = a \cdot r^2 \quad \text{--- Eq ①}$$

and also given that 6th term is 192 i.e. $a_6 = 192$

$$a_n = a \cdot r^{n-1}$$

$$192 = a \cdot r^{6-1}$$

$$192 = a \cdot r^5 \quad \text{--- Eq ②}$$

Divide Eq. ② by Eq. ①

$$\frac{a \cdot r^5}{a \cdot r^2} = \frac{192}{24}$$

$$r^3 = 8$$

$$r^3 = 2^3$$

When powers are equal, bases are also equal

$$\therefore r = 2$$

10th term of Geometric progression.

$$a_n = a \cdot r^{n-1}$$

$$a_{10} = a \cdot r^{10-1}$$

$$= a \cdot r^9$$

To know the value of 'a' substitute 'r' value in any Equation.

$$ar^2 = 24$$

$$a \cdot 2^2 = 24$$

$$a \cdot 4 = 24$$

$$a = \frac{24}{4}$$

$$a = 6$$

$$a_{10} = a \cdot r^9$$

$$= 6 \cdot 2^9$$

$$= 6 \cdot 512$$

$$= 3072$$

4) $x, x+2, x+6$ are consecutive terms of Geometric Progression, then find the value of x
Given series of Geometric Progression.

$$x, x+2, x+6$$

As we know that the ratio between any two terms are equal. i.e. $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

$$\frac{x+2}{x} = \frac{x+6}{x+2}$$

$$(x+2)(x+2) = x(x+6)$$

$$x^2 + 2x + 2x + 4 = x^2 + 6x$$

$$4x + 4 = 6x$$

$$4x - 6x = -4$$

$$-2x = -4$$

$$x = \frac{-4}{-2}$$

$$x = 2$$

$$6x - 4x = 4$$

$$2x = 4$$

$$x = \frac{4}{2}$$

$$x = 2$$

$\therefore$ The value of x is 2.

5) 4, 8, 16 are the given series of G.P. then find next 3 terms of G.P.

Given series of Geometric Progression:

$$4, 8, 16.$$

$$\text{first term } (a) = 4$$

$$\text{Common ratio} = \frac{t_2}{t_1}$$

$$r = 8/4$$

$$r = 2$$

We know that the common series of Geometric Progression are, $a, ar, ar^2, ar^3, ar^4, ar^5$

$$4^{\text{th}} \text{ term} = a \cdot r^3 = 4 \cdot (2)^3 = 4 \times 8 = 32$$

$$5^{\text{th}} \text{ term} = a \cdot r^4 = 4 \cdot (2)^4 = 4 \times 16 = 64$$

$$6^{\text{th}} \text{ term} = a \cdot r^5 = 4 \cdot (2)^5 = 4 \times 32 = 128$$

The next three terms of Geometric progression are 32, 64, 128.

6) Find the sum of first 8 terms of Geometric Progression 3, 6, 12, 24, ...

Given series of Geometric progression

$$3, 6, 12, 24, \dots$$

$$\text{first term } a = 3, t_2 = 6, t_1 = 3$$

$$\text{Common ratio } r = \frac{t_2}{t_1}$$

$$r = 6/3$$

$$r = 2$$

Sum of 8 terms,

$$\text{Sum of } n \text{ terms } S_n = \frac{a \cdot r^n - a}{r - 1}$$

$$S_8 = \frac{3 \cdot 2^{n-1}}{n-1}$$

$$S_8 = \frac{3 \cdot 2^{8-1}}{8-1}$$

$$S_8 = \frac{3 \cdot 2^7}{7}$$

$$= \frac{384}{7}$$

∴ The sum of 8 terms is $\frac{384}{7}$

⑦ If first term is 1 & common ratio is 2, then find sum of first 20 terms in G.P.

Given first term $(a) = 1$

Common ratio $r = 2$

sum of n term is $S_n = \frac{a \cdot r^n - a}{r - 1}$

sum of 20 terms is $S_{20} = \frac{a \cdot r^{20} - a}{r - 1}$

$$= \frac{a \cdot r^{19}}{r - 1}$$

$$S_{20} = \frac{1 \cdot (2)^{19}}{2 - 1}$$

$$= \frac{(2)^{19}}{1}$$

∴ sum of 20 terms is $\frac{(2)^{19}}{1}$

* find the 10th and 20th term of the Geometric Progression with the first term is 3 and common ratio is 2.

Given that first term $(a) = 3$

Common ratio $r = 2$.

n^{th} term of G.P is

$$a_n = a \cdot r^{n-1}$$

10th term of G.P is $a_{10} = a \cdot r^{10-1}$
 $= a \cdot r^9$
 $= 3 \cdot (2)^9$

20th term of G.P is $a_{20} = a \cdot r^{20-1}$
 $= a \cdot r^{19}$
 $= 3 \cdot (2)^{19}$

Harmonic Progression.

Harmonic Progression is the reciprocal to the Arithmetic progression i.e., $\frac{1}{\text{A.P.}}$

$a_1, a_2, a_3 \dots$ are in ~~Arithmetic~~ Harmonic progression then,

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ are in Arithmetic progression.

$\therefore$ The n^{th} term of Arithmetic progression is

$$a_n = a + (n-1)d$$

The n^{th} term of Harmonic progression is

$$a_n = \frac{1}{a + (n-1)d}$$

$\therefore \frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in Harmonic progression the n^{th} term

Given series of harmonic progression is

$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in H.P then $1, 3, 5, 7, \dots$ are in Arithmetic progression.

• first term = 1

Common difference $(d) = t_2 - t_1$

$$d = 3 - 1$$

$$d = 2$$

n^{th} term of Arithmetic Progression

$$a_n = a + (n-1)d$$

$$a_{99} = a + (99-1)d$$

$$= a + 98d$$

$$= 1 + 98(2)$$

$$= 1 + 196$$

$$= 197.$$

The 99th term of Harmonic Progression is $\frac{1}{197}$.

① Insert 4 harmonic means between $\frac{1}{2}$ and $\frac{1}{42}$.
we know that

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \frac{1}{a+4d}, \frac{1}{a+5d} \text{ are in H.P.}$$

then $a, a+d, a+2d, a+3d, a+4d, a+5d$ are in A.P.

Given that

and also given that

$$\frac{1}{a} = \frac{1}{2}$$

$$\text{then } a = 2$$

$$\frac{1}{a+5d} = \frac{1}{42}$$

$$a+5d = 42$$

$$12+5d = 42$$

$$5d = 42 - 12$$

$$d = \frac{30}{5}$$

$$d = 6.$$

$$2^{\text{nd}} \text{ term} = \frac{1}{a+d} = \frac{1}{12+6} = \frac{1}{18} = \frac{1}{18}$$

$$3^{\text{rd}} \text{ term} = \frac{1}{a+2d} = \frac{1}{12+2(6)} = \frac{1}{12+12} = \frac{1}{24}$$

$$4^{\text{th}} \text{ term} = \frac{1}{a+3d} = \frac{1}{12+3(6)} = \frac{1}{12+18} = \frac{1}{30}$$

$$5^{\text{th}} \text{ term} = \frac{1}{a+4d} = \frac{1}{12+4(6)} = \frac{1}{12+24} = \frac{1}{36}$$

$\therefore$ The four harmonic means are $\frac{1}{18}, \frac{1}{24}, \frac{1}{30}, \frac{1}{36}$.

Unit - IV
Vectors and Matrices

Definition:

A rectangular array of numbers taken from field F is known as Matrices. Horizontal arrangement of elements is known as Row and vertical arrangement is known as a "columns".

In Matrices if they are m rows and n columns it is called $m \times n \rightarrow (m \text{ by } n)$ where $m \times n$ is called as order or size of the matrix

Ex: $A = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 2 \\ 1 & 4 & 6 \end{bmatrix}_{3 \times 3}$ $B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}_{2 \times 3}$

The size of Matrix is 3×3

The size of Matrix is 2×3

Note: The compact form of (notation) above the matrix is denoted by $A = [a_{ij}]_{m \times n}$

Where ' i ' indicates particular row value for that element

Where ' j ' indicates particular column value for that element

Ex: $A = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 2 \\ 1 & 4 & 6 \end{bmatrix}_{3 \times 3}$

The notation of Matrix A is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$

This is called notation

Types of Matrices:

* Rectangular Matrix:

A matrix of order $m \times n$ is called as rectangular matrix

Ex: $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}_{2 \times 3}$ $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$

* Square Matrix:

A matrix in which the number of rows is equal to number of columns then the matrix is called square matrix. $m=n$

Ex: $A = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 2 \\ 1 & 4 & 6 \end{bmatrix}_{3 \times 3}$

Ex: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$

* Principal Diagonal:

In a square matrix all those elements a_{ij} for which $[i=j]$ are called elements of principal Diagonal

Ex: $A = \begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 2 \\ 1 & 4 & 6 \end{bmatrix}$

notation of matrix:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$

$\therefore 2, 4, 6$ are the principal Diagonal

* Diagonal Matrix:

A square matrix in which all the elements except principal diagonal elements are "zero" then the matrix called Diagonal Matrix.

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}_{3 \times 3}$

* Scalar Matrix:

A diagonal Matrix in which all the principal diagonal elements are equal is called 'scalar matrix'

Ex: $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3 \times 3}$

6. Unit Matrix or Identity Matrix:

A square matrix in which all the principal diagonal elements are equal to one and all other elements are zero then the matrix is called unit matrix.

$$\text{Ex: } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

7. Null Matrix or Zero Matrix:

A matrix in which all the elements are zero is called Null matrix.

$$\text{Ex: } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$$

14/3/21 Transpose of a Matrix:

The Matrix obtained by changing the rows into columns or columns into rows is called a Transpose of a Matrix.

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$$

$$\text{Transpose of a matrix } A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2 \times 2}$$

1. Matrix Addition:

$$\text{If } A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 9 & 2 \end{bmatrix} \text{ then find } A+B$$

Sol: Given Matrices

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 1 & 4 \\ 9 & 2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+4 \\ 6+9 & 8+2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 10 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 7 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$ then prove that Matrix addition is associative. (4)

Sol: Given Matrices

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 7 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

The Associative Property of Matrices is

$$(A+B)+C = A+(B+C)$$

$$A+B = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 16 & 3 \end{bmatrix}$$

L.H.S

$$(A+B)+C = \begin{bmatrix} 3 & 7 \\ 16 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 16 & 7 \end{bmatrix}$$

R.H.S

$$(B+C) = \begin{bmatrix} 2 & 4 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 5 \end{bmatrix}$$

$$(B+C)+A = \begin{bmatrix} 5 & 6 \\ 7 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 16 & 7 \end{bmatrix}$$

R.H.S

$$L.H.S = R.H.S$$

$\therefore$ Hence the Matrices addition is associative.

3. $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ where $A+B=X$

then find x_1, x_2, x_3 and x_4 .

Sol: Given

$$A+B=X$$

$$\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+2 & 3+1 \\ 4+3 & 2+(-5) \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

If order of Matrices are same in both sides then the corresponding values can be equal.

$$x_1 = 1, x_2 = 4, x_3 = 7, x_4 = -3$$

Subtraction of Matrices:

1. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 4 \\ 6 & 7 & 4 \end{bmatrix}$ then find $A - B$

Sol:

Given

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 4 \\ 6 & 7 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 4 \\ 6 & 7 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0-0 & 1-1 & 2-4 \\ 2-6 & 3-7 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -4 & -4 & 0 \end{bmatrix}$$

$$\therefore A - B = \begin{bmatrix} 0 & 0 & -2 \\ -4 & -4 & 0 \end{bmatrix}$$

Multiplication of Matrices:

If any two Matrices A and B to be multiplied ($A \times B$) is possible only if it satisfies the following condition:

The number of Rows of 1st Matrix (A) = The Number of columns of 2nd Matrix (B), then only the Matrix multiplication is possible.

Syntax of Matrix Multiplication:

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ then $A \times B =$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A \times B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

2x2

1. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$ then find $A \times B$

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (1 \times 4) + (2 \times 1) & (1 \times 1) + (2 \times 2) \\ (2 \times 4) + (3 \times 1) & (2 \times 1) + (3 \times 2) \end{bmatrix} = \begin{bmatrix} 4+2 & 1+4 \\ 8+3 & 2+6 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 6 & 5 \\ 11 & 8 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 9 \\ 9 & 9 \end{bmatrix}$ then find $A \times B$ and $B \times A$

Sol: Given $A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ 9 & 9 \end{bmatrix}$

$$A \times B = \begin{bmatrix} (1 \times 1) + (6 \times 9) & (1 \times 9) + (6 \times 9) \\ (1 \times 9) + (4 \times 9) & (1 \times 9) + (4 \times 9) \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1+54 & 9+54 \\ 1+36 & 9+36 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 55 & 63 \\ 37 & 45 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 9 \\ 9 & 9 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} (1 \times 1) + (9 \times 1) & (1 \times 6) + (9 \times 4) \\ (9 \times 1) + (9 \times 1) & (9 \times 6) + (9 \times 4) \end{bmatrix}$$

$$B \times A = \begin{bmatrix} 1+9 & 6+36 \\ 9+9 & 54+36 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} 10 & 42 \\ 18 & 90 \end{bmatrix}$$

3. If $A = \begin{bmatrix} -1 & 3 \\ -6 & 9 \end{bmatrix}$ then find A^2

Sol: Given that

$$A = \begin{bmatrix} -1 & 3 \\ -6 & 9 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} -1 & 3 \\ -6 & 9 \end{bmatrix} \times \begin{bmatrix} -1 & 3 \\ -6 & 9 \end{bmatrix} = \begin{bmatrix} (-1 \times -1) + (3 \times -6) & (-1 \times 3) + (3 \times 9) \\ (-6 \times -1) + (9 \times -6) & (-6 \times 3) + (9 \times 9) \end{bmatrix}$$

$$= \begin{bmatrix} 1-18 & -3+27 \\ 6-54 & -18+81 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 17 & 24 \\ -48 & 13 \end{bmatrix}$$

Imp

4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then show that $A^2 - 4A = 5I$

Given that

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = A \times A \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1 \times 1) + (2 \times 2) + (2 \times 2) & (1 \times 2) + (2 \times 1) + (2 \times 2) & (1 \times 2) + (2 \times 2) + (2 \times 1) \\ (2 \times 1) + (1 \times 2) + (2 \times 2) & (2 \times 2) + (1 \times 1) + (2 \times 2) & (2 \times 2) + (1 \times 2) + (2 \times 1) \\ (2 \times 1) + (2 \times 2) + (1 \times 2) & (2 \times 2) + (2 \times 1) + (1 \times 2) & (2 \times 2) + (2 \times 2) + (1 \times 1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1+4+4) & (2+2+4) & (2+4+2) \\ (2+2+4) & (4+1+4) & (4+2+2) \\ (2+4+2) & (4+2+2) & (4+4+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$A \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$A^2 - 4A$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 9-4 & 8-8 & 8-8 \\ 8-8 & 9-4 & 8-8 \\ 8-8 & 8-8 & 9-4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5I$$

$$\Rightarrow 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 5(I)$$

$$\Rightarrow 5I$$

$\therefore$ Hence proved

$$A^2 - 4A = 5I$$

5. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find $A + A^T$

Given that

$$A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$ then find $A \cdot B$ and

Sol: Given that

$$A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix} \quad B^T = \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (7 \times -2) + (-2 \times -1) & (7 \times 4) + (-2 \times 2) & (7 \times -1) + (-2 \times 0) \\ (-1 \times -2) + (2 \times -1) & (-1 \times 4) + (2 \times 2) & (-1 \times -1) + (2 \times 0) \\ (5 \times -2) + (3 \times -1) & (5 \times 4) + (3 \times 2) & (5 \times -1) + (3 \times 0) \end{bmatrix}$$

$$A \cdot B^T \Rightarrow \begin{bmatrix} -12 & 24 & -7 \\ 0 & 0 & 1 \\ -13 & 26 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$B \cdot A^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (-2 \times 7) + (-1 \times -2) & (-2 \times -1) + (-1 \times 2) & (-2 \times 5) + (-1 \times 3) \\ (4 \times 7) + (2 \times -2) & (4 \times -1) + (2 \times 2) & (4 \times 5) + (2 \times 3) \\ (-1 \times 7) + (0 \times -2) & (-1 \times -1) + (0 \times 2) & (-1 \times 5) + (0 \times 3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (-14) + (2) & (2) + (-2) & (-10) + (-3) \\ (28) + (-4) & (-4) + (4) & (20) + (6) \\ (-7) + (0) & (1) + (0) & (-5) + (0) \end{bmatrix} = \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix} = A \cdot B^T$$

$$A \cdot B^T = \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ Find A^T , B^T , $(AB)^T$.

$$A \cdot B^T$$

Sol: Given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}, B^T = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} (-14) + (2) & (2) + (-2) & (-10) + (-3) \\ (28) + (-4) & (-4) + (4) & (20) + (6) \\ (-7) + (0) & (1) + (0) & (-5) + (0) \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T \times \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad (10)$$

$$\Rightarrow \begin{bmatrix} (1 \times 9) + (2 \times 6) + (3 \times 3) & (1 \times 8) + (2 \times 5) + (3 \times 2) & (1 \times 7) + (2 \times 4) + (3 \times 1) \\ (4 \times 9) + (5 \times 6) + (6 \times 3) & (4 \times 8) + (5 \times 5) + (6 \times 2) & (4 \times 7) + (5 \times 4) + (6 \times 1) \\ (7 \times 9) + (8 \times 6) + (9 \times 3) & (7 \times 8) + (8 \times 5) + (9 \times 2) & (7 \times 7) + (8 \times 4) + (9 \times 1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (9 + 12 + 9) & (8 + 10 + 6) & (7 + 8 + 3) \\ (36 + 30 + 18) & (32 + 25 + 12) & (28 + 20 + 6) \\ (63 + 48 + 27) & (56 + 40 + 18) & (49 + 32 + 9) \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{bmatrix}$$

$$A \cdot B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times 9 + 2 \times 8 + 3 \times 7) & (1 \times 6 + 2 \times 5 + 3 \times 4) & (1 \times 3 + 2 \times 2 + 3 \times 1) \\ (4 \times 9 + 5 \times 8 + 6 \times 7) & (4 \times 6 + 5 \times 5 + 6 \times 4) & (4 \times 3 + 5 \times 2 + 6 \times 1) \\ (7 \times 9 + 8 \times 8 + 9 \times 7) & (7 \times 6 + 8 \times 5 + 9 \times 4) & (7 \times 3 + 8 \times 2 + 9 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} (9 + 16 + 21) & (6 + 10 + 12) & (3 + 4 + 3) \\ (36 + 40 + 42) & (24 + 25 + 24) & (12 + 10 + 6) \\ (63 + 64 + 63) & (42 + 40 + 36) & (21 + 16 + 9) \end{bmatrix}$$

$$= \begin{bmatrix} 46 & 28 & 10 \\ 118 & 73 & 28 \\ 190 & 118 & 46 \end{bmatrix}$$

$$\begin{bmatrix} 46 & 28 & 10 \\ 118 & 73 & 28 \\ 190 & 118 & 46 \end{bmatrix}$$

21 16 19 46

(...)

Determination of a matrix (or) ^{Let} Data of a matrix:

Let A and B ^{is} are any square matrices then the determination of a matrix is denoted by.

$|A|$. The calculation of a determinant of matrix A is as follows.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } |A| = ad - bc$$

Cofactors of a matrix :

The cofactors of an element in the i^{th} row and j^{th} column of any given square matrix is defined as with its minor is multiplied with $(-1)^{i+j}$ i.e., cofactor $= (-1)^{i+j}$ M.E (Minor elements).

$$\text{If } A = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \text{ then find cofactors of } A.$$

Given Matrix

$$A = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

To find co-factors we use $(-1)^{i+j}$ M.E

$$\begin{aligned} \text{co-factor of } 1 &= (-1)^{1+1} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (-1)^2 [(0 \times 1) - (0 \times 0)] \\ &= 1(0 - 0) \\ &= 1(0) - 0 \end{aligned}$$

$$\begin{aligned} \text{cofactor of } 0 &= (-1)^{1+2} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = (-1)^3 (2 - 0) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{cofactor of } 8 &= (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = (-1)^4 [(2 \times 0) - (3 \times 0)] \\ &= 1(0) = 0 \end{aligned}$$

$$\text{cofactor of } 2(-1)^{2+1} = \begin{vmatrix} 0 & 8 \\ 0 & 1 \end{vmatrix} = (-1)^3 [(0 \times 1) - (3 \times 0)]$$

$$\text{cofactor of } 0(-1)^{2+2} = \begin{vmatrix} 1 & 8 \\ 3 & 1 \end{vmatrix} = (-1)^4 [(1 \times 1) - (3 \times 8)]$$

$$\text{cofactor of } 0(-1)^{2+3} = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = (-1)^5 [(2 \times 0) - (3 \times 0)]$$

$$\text{cofactor of } 3(-1)^{3+1} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (-1)^4 [(0 \times 1) - (0 \times 0)]$$

$$\text{cofactor of } 0(-1)^{3+2} = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = (-1)^5 [(1 \times 0) - (2 \times 0)]$$

$$\text{cofactor of } 1(-1)^{3+3} = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = (-1)^6 [(1 \times 0) - (2 \times 0)]$$

$$A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -23 & 0 \\ 0 & 16 & 0 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ then find $|A|$

Sol:

Given that

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$$

$$|A| = 1[(-1 \times 6) - (2 \times 5)] - 0[(3 \times 6) - (2 \times 4)] + (-2)[(3 \times 5) - (-1 \times 4)]$$

$$|A| = 1[-6 - 10] - 0 - 2(15 - 4)$$

$$|A| = -16 - 2(19)$$

$$|A| = -16 - 38$$

$$|A| = -54$$

Previous
Q.P
problem

If $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & -1 & 3 \\ 1 & -5 & 6 \end{bmatrix}$ then find $|A|$

Given Matrix

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & -1 & 3 \\ 1 & -5 & 6 \end{bmatrix}$$

$$\begin{aligned} |A| &= 3 [(-1 \times 6) - (3 \times -5)] - 2 [(4 \times 6) - (3 \times 1)] + 0 [(4 \times -5) - (1 \times 1)] \\ &= 3 [-6 + 15] - 2 [24 - 3] + 0 [-20 + 1] \\ &= 3 (9) - 2 (21) + 0 (-19) \\ &= 27 - 42 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$$

$$|A| = -15$$

$$(1 - 31) 0 - 0 - (01 - 0 -) 1 = |A|$$

$$(11) 0 - 01 - = |A|$$

$$00 - 01 - = |A|$$

Adjoint of a Matrix:

A Adjoint matrix is a matrix which is the transpose of the co-factor matrix which is denoted by Adj(A)

Find the Adjoint matrix of $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

Sol: Given Matrix $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

cofactor of (2) = 8

cofactor of (4) = 6

cofactor of (6) = 4

cofactor of (8) = 2

cofactor of Matrix 'A' is $= \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}$

$\therefore \text{Adj}(A) = \begin{bmatrix} 8 & 4 \\ 6 & 2 \end{bmatrix}$

Find the Adjoint of Matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 2 & 0 \end{bmatrix}$

Sol: Given Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 2 & 0 \end{bmatrix}$

co-factor of '1' $= (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 2 & 0 \end{vmatrix} = (1) \begin{vmatrix} 5 & 6 \\ 2 & 0 \end{vmatrix}$

$$= (-1)^1 ((5 \times 0) - (6 \times 2)) = -12$$

$$= -12$$

$$= -12$$

cofactor of '2' $= (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 4 & 0 \end{vmatrix}$

$$= (-1)^3 ((4 \times 0) - (6 \times 4))$$

$$= -1 (0 - 24)$$

$$= 24$$

$$\text{cofactor of } 3 \Rightarrow (-1)^{3+1} \Rightarrow \begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix} \Rightarrow (-1)^4 \Rightarrow ((4 \times 2) - (5 \times 4))$$

$$\begin{bmatrix} -12 \\ 6 \\ -3 \end{bmatrix} = A \text{ cofactor } \Rightarrow 70 = 3000 \Rightarrow (8-20) \Rightarrow -12$$

$$\text{cofactor of } 4 \Rightarrow (-1)^{2+1} \Rightarrow \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} \Rightarrow (-1)^3 \Rightarrow ((2 \times 0) - (3 \times 2))$$

$$\begin{bmatrix} -12 \\ 6 \\ -3 \end{bmatrix} = (-1) \Rightarrow (0-6) \Rightarrow -6$$

$$\text{cofactor of } 5 \Rightarrow (-1)^{2+2} \Rightarrow \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} \Rightarrow (-1)^4 \Rightarrow ((1 \times 0) - (3 \times 4))$$

$$\Rightarrow 1 \Rightarrow (0-12) \Rightarrow -12$$

$$\text{cofactor of } 6 \Rightarrow (-1)^{2+3} \Rightarrow \begin{vmatrix} 4 & 5 \\ 4 & 2 \end{vmatrix} \Rightarrow (-1)^5 \Rightarrow ((4 \times 2) - (5 \times 4))$$

$$\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = (-1) = (1 \times 2) - (4 \times 2) \Rightarrow (-1) \Rightarrow (8-20) \Rightarrow -12$$

$$= (2) - (8) \Rightarrow 6 \Rightarrow (-1) \Rightarrow -2 \Rightarrow -12$$

$$\text{cofactor of } 4 \Rightarrow (-1)^{3+1} \Rightarrow \begin{vmatrix} 5 & 6 \\ 2 & 3 \\ 2 & 0 \end{vmatrix} \Rightarrow (-1)^4 \Rightarrow ((2 \times 6) - (3 \times 5))$$

$$\Rightarrow 1 \Rightarrow (12-15) \Rightarrow -3$$

$$\text{cofactor of } 2 \Rightarrow (-1)^{3+2} \Rightarrow \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \Rightarrow (-1)^5 \Rightarrow ((1 \times 6) - (3 \times 4))$$

$$= (1) \times 6 = 6 \Rightarrow (-1) \Rightarrow (6-12) \Rightarrow -6$$

$$\text{cofactor of } 0 \Rightarrow (-1)^{3+3} \Rightarrow \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \Rightarrow (-1)^6 \Rightarrow ((1 \times 5) - (2 \times 4))$$

$$\Rightarrow -1 \Rightarrow (5-8) \Rightarrow -3$$

$$\text{co-factor of } A = \begin{bmatrix} -12 & 24 & -12 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} -12 & 6 & -3 \\ 24 & -12 & 6 \\ -12 & 6 & -3 \end{bmatrix}$$

Inverse of a Matrix is denoted by A^{-1}

Find the inverse of a Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

Given Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \Rightarrow (1 \times 4) - (2 \times 3) \Rightarrow 4 - 6 \Rightarrow -2$$

$$|A| = -2$$

$$\text{C.F.}(A) = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/-2 & 3/-2 \\ 2/-2 & 1/-2 \end{bmatrix} = \begin{bmatrix} -2 & -1.5 \\ -1 & -0.5 \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

then find A^{-1}

odd numbers are always
even no.s are always

$$\begin{aligned} |A| &= \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3[(-3 \times 1) - (-1 \times 2)] - 1[(2 \times 1) - (-1 \times 1)] - 2[(2 \times 2) - (-3 \times 1)] \\ &= 3[(-3 + 2) - (-1)] - 1[2 + 1] - 2[4 + 3] \\ &= 3(-1) - 1(3) - 2(7) = -3 - 3 - 14 = -6 - 14 = -20 \end{aligned}$$

Cofactor of 3 = $(-1)^{1+1} \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} = 1[(-3 \times 1) - (-1 \times 2)] = 1[-3 + 2] = -1$

Cofactor of 1 = $(-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -1[(2 \times 1) - (-1 \times 1)] = -1[2 + 1] = -3$

Cofactor of 2 = $(-1)^{1+3} \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 1[(2 \times 2) - (-3 \times 1)] = 1[4 + 3] = 7$

$$\text{cofactor of } 2 = (-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -1 [(1 \times 1) - (2 \times 2)] = -1(1 - 4) = 3$$

$$\text{cofactor of } -3 = (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 [(3 \times 1) - (2 \times 1)] = 1(3 - 2) = 1$$

$$\text{cofactor of } -1 = (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -1 [(3 \times 2) - (1 \times 1)] = -1(6 - 1) = -5$$

$$\text{cofactor of } 1 = (-1)^{3+1} \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 1 [(3 \times -1) - (2 \times 2)] = 1(-3 - 4) = -7$$

$$\text{cofactor of } 2 = (-1)^{3+2} \begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = -1 [(3 \times -3) - (1 \times 2)] = -1(-9 - 2) = 11$$

$$\text{cofactor of } 1 = (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = 1 [(3 \times -3) - (1 \times 2)] = 1(-9 - 2) = -11$$

$$\text{cofactor of } A = \begin{bmatrix} -1 & 3 & 5 \\ 3 & 1 & -7 \\ 5 & 7 & -11 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & 11 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & 11 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} \text{ Find } A^{-1}$$

Given that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix}$$

$$|A| = 1(16 - 0) - 0 + 0$$

$$|A| = 16$$

$$\text{cofactor of } (A) = \begin{bmatrix} 16 & -32 & -2 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} 16 & 0 & 0 \\ -32 & 8 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{16} \begin{bmatrix} 16 & 0 & 0 \\ -32 & 8 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \text{ Find } A^{-1}$$

Given that

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

$$|A| = 2[(0 \times 1) - (1 \times 2)] - 1[(1 \times 1) - (1 \times 2)] + 2[(1 \times 2) - (0 \times 2)]$$

$$= 2[(0 - 2) - 1(1 - 2) + 2(2 - 0)]$$

$$= -4 + 1 + 4 = 1 \Rightarrow |A| = 1$$

$$|A| = 1$$

$$\text{Adj}(A) = \frac{1}{|A|} \text{Adj}(A) \quad \text{or a constant scalar value}$$

$\text{Adj}(A) = \text{Transpose of co-factor Matrix of } A$

$$\text{cofactor of } 2 = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 \text{ (or } 1 - (2 \times 1))$$

$$\text{cofactor of } 1 = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -1 \text{ (or } -(2 \times 2 - 1 \times 1))$$

$$\text{cofactor of } 2 = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2 \text{ (or } -(1 \times 2 - 1 \times 0))$$

$$\text{cofactor of } 1 = (-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \text{ (or } -(1 \times 1 - 2 \times 2))$$

$$\text{cofactor of } 0 = (-1)^{2+2} \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -2 \text{ (or } -(2 \times 2 - 2 \times 1))$$

$$\text{cofactor of } 1 = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 0 \text{ (or } -(2 \times 2 - 2 \times 1))$$

$$\text{cofactor of } 2 = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \text{ (or } 1 \times 1 - 2 \times 0)$$

$$\text{cofactor of } 2 = (-1)^{3+2} \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = 2 \text{ (or } -(2 \times 0 - 2 \times 1))$$

$$\text{cofactor of } 1 = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \text{ (or } -(2 \times 0 - 1 \times 1))$$

$$\text{cofactor of } A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} -2 & 3 & 1 \\ 1 & -2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

Matrix inverse Method using Linear equations to obtain the values of x .

Procedure:

If $a_1x + a_2y + a_3z$

If linear equations are

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

If the given linear equations are

step 1:

convert the given linear equations into augmented Matrix form i.e., $AX = B$ where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

step 2:

$$\text{Now find } A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

step 3:

To obtain values of x (variables x, y, z) we have to find

$$X = A^{-1} \cdot B$$

Find the values of x, y, z by using Matrix inverse Method for the given system

$$3x + 4y + 5z = 18$$

$$2x - y + 8z = 13$$

$$5x - 2y + 7z = 20$$

Sol:

Given system of equations

$$3x + 4y + 5z = 18$$

$$2x - y + 8z = 13$$

$$5x - 2y + 7z = 20$$

convert the given system into augmented matrix form

i.e., $AX=B$

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} \Rightarrow 3[(-1 \times 7) - (8 \times -2)] - 4(2 \times 7) - (8 \times 5) + 5(2 \times -2) + (-1 \times 5)$$

$$\Rightarrow 3[-7 + 16] - 4[14 - 40] + 5[-4 + 5]$$

$$\Rightarrow 3[9] - 4[-26] + 5[1]$$

$$\Rightarrow 27 + 104 + 5 \Rightarrow |A| = 136$$

cofactor of A $\Rightarrow$

$$3 \Rightarrow (-1)^{1+1} \Rightarrow \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} \Rightarrow (-1)^3 = (-1 \times 7) - (8 \times -2) = -7 + 16 = 9$$

$$4 \Rightarrow (-1)^{1+2} \Rightarrow \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} \Rightarrow (-1)^3 = (2 \times 7) - (8 \times 5) = 14 - 40 = -26$$

$$5 \Rightarrow (-1)^{1+3} \Rightarrow \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} \Rightarrow (-1)^3 = (2 \times -2) - (-1 \times 5) = -4 + 5 = 1$$

$$2 \Rightarrow (-1)^{2+1} \Rightarrow \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} \Rightarrow (-1)^3 = (4 \times 7) - (5 \times -2) = 28 + 10 = 38$$

$$-1 \Rightarrow (-1)^{2+2} \Rightarrow \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} \Rightarrow (-1)^4 = (3 \times 7) - (5 \times 5) = 21 - 25 = -4$$

$$8 \Rightarrow (-1)^{2+3} \Rightarrow \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} \Rightarrow (-1)^5 = (3 \times -2) - (4 \times 5) = -6 - 20 = -26$$

$$\Rightarrow 5(-1) = \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} = (4 \times 8) - (5 \times -1) = (32) - (-5) = 37$$

$$\Rightarrow -2(-1) = \begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} = (3 \times 8) - (5 \times 2) = (24) - (10) = 14$$

$$\Rightarrow 7(-1) = \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = (3 \times -1) - (4 \times 2) = (-3) - (8) = -11$$

$$= \begin{bmatrix} 37 & 14 & -11 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

Cofactors of A = $\begin{bmatrix} 37 & 14 & -11 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

$$\text{Adj}(A) = \begin{bmatrix} 37 & 26 & 1 \\ 14 & -4 & 26 \\ -11 & -14 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{136} \begin{bmatrix} 37 & 26 & 1 \\ 14 & -4 & 26 \\ -11 & -14 & 1 \end{bmatrix}$$

Step 3: $X = A^{-1}B$

$$\Rightarrow \frac{1}{136} \begin{bmatrix} 37 & 26 & 1 \\ 14 & -4 & 26 \\ -11 & -14 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$\Rightarrow \frac{1}{136} \begin{bmatrix} (37 \times 18) + (26 \times 13) + (1 \times 20) \\ (14 \times 18) + (-4 \times 13) + (26 \times 20) \\ (-11 \times 18) + (-14 \times 13) + (1 \times 20) \end{bmatrix}$$

$$\Rightarrow \frac{1}{136} \begin{bmatrix} 662 & -494 & 740 \\ 468 & -52 & -280 \\ 18 & 338 & -220 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{136} \begin{bmatrix} 662 \\ 468 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{408}{136} \\ \frac{136}{136} \\ \frac{136}{136} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore x=3, y=1, z=1$$

Verification:

Substitute $x=3, y=1, z=1$ in any equation in the given system

$$\begin{aligned} \text{1st eq } 3x + 4y + 5z &= 18 \\ 3(3) + 4(1) + 5(1) &= 18 \\ 9 + 4 + 5 &= 18 \end{aligned}$$

$$\boxed{18=18}$$

Therefore the obtained values 'x' are correct and hence verified.

$$5x - 6y + 4z = 15$$

$$7x - 4y - 3z = 19$$

$$2x - y + 6z = 46$$

Given system of equations

$$5x - 6y + 4z = 15$$

$$7x - 4y - 3z = 19$$

$$2x - y + 6z = 46$$

Convert the given systems into augmented matrix form i.e., $\boxed{AX=B}$

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & -4 & -3 \\ 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & -4 & -3 \\ 2 & -1 & 6 \end{vmatrix}$$

$$|A| = 5[(-4 \times 6) - (-3 \times -1)] - [(-6)(7 \times 6 - (-3 \times 2))] + 4[(7 \times -1) - (-4 \times 2)]$$

$$= 5(-27) + 6(48) + 4(1)$$

$$= -135 + 288 + 4$$

$$|A| = 157$$

Co-factors of $\begin{vmatrix} 5 & -6 & 4 \\ 7 & -4 & -3 \\ 2 & -1 & 6 \end{vmatrix}$

co-factor of 5 = $(-1)^{1+1} \begin{vmatrix} -4 & -3 \\ -1 & 6 \end{vmatrix} = -27$

co-factor of (-6) = $(-1)^{1+2} \begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -48$

co-factor of 4 = $(-1)^{1+3} \begin{vmatrix} 7 & -4 \\ 2 & -1 \end{vmatrix} = -5$

co-factor of 7 = $(-1)^{2+1} \begin{vmatrix} -6 & 4 \\ -1 & 6 \end{vmatrix} = 32$

co-factor of -4 = $(-1)^{2+2} \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 22$

co-factor of -3 = $(-1)^{2+3} \begin{vmatrix} 5 & -6 \\ 2 & -1 \end{vmatrix} = -7$

co-factor of 2 = $(-1)^{3+1} \begin{vmatrix} -6 & 4 \\ -4 & -3 \end{vmatrix} = 34$

co-factor of -1 = $(-1)^{3+2} \begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = 43$

co-factor of 6 = $(-1)^{3+3} \begin{vmatrix} 5 & -6 \\ 7 & -4 \end{vmatrix} = 22$

co-factor of $A = \begin{bmatrix} -24 & -48 & 34 \\ 32 & 22 & -7 \\ 34 & 43 & 22 \end{bmatrix}$

$\text{Adj}(A) = \begin{bmatrix} -24 & 32 & 34 \\ -48 & 22 & 43 \\ 1 & -7 & 22 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{Adj } A$

$= \frac{1}{157} \begin{bmatrix} -24 & 32 & 34 \\ -48 & 22 & 43 \\ 1 & -7 & 22 \end{bmatrix}$

Sp: 3 $X = A^{-1}B$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{157} \begin{bmatrix} -24 & 32 & 34 \\ -48 & 22 & 43 \\ 1 & -7 & 22 \end{bmatrix} \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

$= \frac{1}{157} \begin{bmatrix} -405 + 608 + 1564 \\ -720 + 418 + 1978 \\ 15 - 123 + 1012 \end{bmatrix}$

$= \frac{1}{157} \begin{bmatrix} 1767 \\ 1676 \\ 394 \end{bmatrix} = \begin{bmatrix} 11.25 \\ 10.68 \\ 5.69 \end{bmatrix}$

$x = 11.25, y = 10.68, z = 5.69$

Verification.

sub the values in any equation

$5x - 6y + 4z = 15$

$5(11.25) - 6(10.68) + 4(5.69) = 15$

$56.25 - 64.08 + 22.76 = 15$

$14.93 = 15$

Hence Verified.

Cramer's Rule

Solve the Equations

$$2x + y - z = 1$$

$$x - y + z = 2$$

$$5x + 5y - 4z = 3 \text{ by using Cramer's Rule.}$$

Given

system of Equations

$$2x + y - z = 1$$

$$x - y + z = 2$$

$$5x + 5y - 4z = 3 \text{ by using } \Rightarrow$$

Convert the given equations into augmented matrix form $AX=B$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 5 & 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 5 & 5 & -4 \end{vmatrix}$$

In Cramer's Rule we find $\frac{\Delta_i}{\Delta}$

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

$$z = \frac{\Delta_3}{\Delta}$$

To find the values of Matrix 'x'.

$$= 2 \begin{vmatrix} -1 & 1 \\ 5 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 5 & -4 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 5 & 5 \end{vmatrix}$$

$$= 2(4-5) - 1(-4-5) - 1(5+5)$$

$$= 2(-1) - 1(-9) - 1(10)$$

$$= -2 + 9 - 10 \quad \boxed{\Delta = 3}$$

$$\Delta_1 = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 5 & 4 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$$

$$\Rightarrow 1(5-3) - 1(12-3) + 1(10-3) = 3$$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 5 & 3 & -4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix}$$

$$\Rightarrow 2(8-3) - 1(4-5) + 1(3-10) = 2(-11) - 1(-9) +$$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 5 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 2 \\ 5 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 5 & 5 \end{vmatrix}$$

$$\Rightarrow -6 + 1(7)$$

$$= 2(-3-10) + 1(3-10) + 1(5+5)$$

$$= 2(-13) + 1(-7) + 1(10) \Rightarrow -26 + 7 + 10$$

$$x = \frac{\Delta_1}{\Delta} \Rightarrow \frac{-3}{-3} = 1$$

$$y = \frac{\Delta_2}{\Delta} \Rightarrow \frac{-6}{-3} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-9}{-3} = 3$$

Verification:

Substitute obtained values $x=1, y=2, z=3$ in any given equation

$$1^{st} \text{ Eq: } 2x + y - z = 1$$

$$2(1) + 2 - 3 = 1$$

$$2 + 2 - 3 = 1$$

$$4 - 3 = 1$$

$$1 = 1$$

$\therefore$ The obtained values of x, y, z are verified and correct

Solve the system of Equations

$$3x - 2y + z = 15$$

$$-x + y + 2z = -10$$

$$x - y - 4z = 14 \quad \text{By Crammer's Rule}$$

Given

system of Equations

$$3x - 2y + z = 15$$

$$-x + y + 2z = -10$$

$$x - y - 4z = 14$$

convert the given system of Equations in the augmented form $[A]x = B$

$$\begin{bmatrix} 3 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 14 \end{bmatrix}$$

In Cramer's Rule we find

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

$$z = \frac{\Delta_3}{\Delta}$$

To find the values of Matrix of x

$$\Delta_1 = \begin{vmatrix} 15 & -2 & 1 \\ -10 & 1 & 2 \\ 14 & -1 & -4 \end{vmatrix} = 15 \begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\Rightarrow 3[(1 \times -4) - (2 \times -1)] + 2[(-1 \times -4) - (2 \times 1)] - 1[(-1 \times 1) - (1 \times 1)]$$

$$\Rightarrow 3[(-4) - (-2)] + 2[(-4) - (2)] - 1[(-1) - (1)]$$

$$\Rightarrow 3(-4 + 2) + 2(-4 - 2) - 1(-1 - 1)$$

$$\Rightarrow 3(-2) + 4$$

$$\Delta_1 \Rightarrow -6 + 4 \Rightarrow -2$$

$$\Delta_1 = \begin{bmatrix} 15 & -2 & 1 \\ -10 & 1 & 2 \\ 14 & -1 & -4 \end{bmatrix}$$

$$\Rightarrow 15 \begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\Rightarrow 15(-4 + 2) + 2(-4 - 2) - 1(-1 - 1)$$

$$\Rightarrow 15(-2) + 2(-4 - 2) - 1(-1 - 1)$$

$$\Rightarrow -30 + 4 - 4 = -30$$

$$\Delta_2 = \begin{bmatrix} 3 & 15 & 1 \\ -1 & -10 & 2 \\ 1 & 14 & -4 \end{bmatrix} \Rightarrow 3 \begin{vmatrix} -10 & 2 \\ 14 & -4 \end{vmatrix} - 15 \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} -1 & -10 \\ 1 & 14 \end{vmatrix}$$

$$\Rightarrow 3(-40 - 28) - 15(-4 - 2) + 1(-14 - 70)$$

$$\Rightarrow 30(12) - 15(-2) + 1(-14 - 70)$$

$$\Rightarrow 360 + 30 - 84 = 306$$

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 15 \\ -1 & 1 & -10 \\ 1 & -1 & 14 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & -10 \\ -1 & 14 \end{vmatrix} + 2 \begin{vmatrix} -1 & -10 \\ 1 & 14 \end{vmatrix} + 15 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 3(14 - 10) + 2(-14 + 10) + 15(1 - 1)$$

$$= 3(4) + 2(-4) + 15(0)$$

$$= 12 - 8 + 0$$

$$\Delta_3 = 4$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-10}{-2} = 5$$

$$y = \frac{\Delta_2}{\Delta} = \frac{2}{-2} = -1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{4}{-2} = -2$$

Substitute x, y, z values in equation $x=5, y=-1, z=-2$ in eq ①

$$3x - 2y + z = 15$$

$$3(5) - 2(-1) + (-2) = 15$$

$$15 + 2 - 2 = 15$$

$$17 - 2 = 15$$

$$\boxed{15 = 15}$$

$\therefore x, y, z$ values are verified and correct.

Rank of a Matrix:

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ then find Rank of a Matrix.

Sol: Given Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ 3×3

$$\Rightarrow |A| = 1 \begin{vmatrix} 7 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 6 & 7 \\ 3 & 2 \end{vmatrix}$$

$$= 1(7-4) - 2(6-6) + 3(12-21)$$

$$\Rightarrow 1(3) - 2(0) + 3(-9)$$

$$\Rightarrow 3 - 0 + 27$$

$$\Rightarrow -24 \neq 0$$

$\therefore$ Hence it is a non-singular Matrix then the rank of $(A) = 3$

2. Find the Rank of a Matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Sol: Given Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$\Rightarrow |A| \Rightarrow 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$\Rightarrow 1(45-48) - 2(36-42) + 3(32-35)$$

$$\Rightarrow 1(3) - 2(-6) + 3(-3)$$

$$\Rightarrow 3 + 6 - 9$$

$$\Rightarrow 9 - 9 = 0$$

$\therefore |A| = 0$, it is a singular Matrix

$\therefore$ Singular Matrix can be converting to the non-singular by elementing any row and column to find the rank

of a matrix
Now, in this we are eliminating 3 rows and 3 columns
the resultant matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_{2 \times 2} \quad |A| = 5 - 8 = -3 \neq 0$$

$\therefore$ The rank of a matrix is 2.

Find the rank of the Matrix

$$A = \begin{bmatrix} 1 & 4 & 3 & 4 \\ 2 & 6 & 7 & 9 \\ 4 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 4 \\ 0 & -2 & 1 & 1 \\ 0 & -10 & -11 & -14 \end{bmatrix}$$

Given that

$$A = \begin{bmatrix} 1 & 4 & 3 & 4 \\ 2 & 6 & 7 & 9 \\ 4 & 2 & 1 & 2 \end{bmatrix}$$

The given Matrix is not a square matrix then we need to convert the given matrix into square matrix by eliminating column.

We perform column operations on any columns. we are subtracting column 4 with column 3

$$C_3 \rightarrow C_4 - C_3$$

$$\begin{bmatrix} 1 & 4 & (4-3) \\ 2 & 6 & (9-7) \\ 4 & 2 & (2-1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 1 \\ 2 & 6 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$|A| = 1 \begin{vmatrix} 6 & 2 \\ 2 & 1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 6 \\ 4 & 2 \end{vmatrix}$$

$$\Rightarrow 1 [(6 \times 1) - (2 \times 2)] - 4 [(2 \times 1) - (2 \times 4)] + 1 [(2 \times 2) - (6 \times 4)]$$

$$\Rightarrow 1 [(6) - (4)] - 4 [(2) - (8)] + 1 [(4) - (24)]$$

$$\Rightarrow 1 (2) - 4 (-6) + 1 (-20)$$

$$\Rightarrow 2 + 24 - 20 \Rightarrow 6 \quad \text{Rank of Matrix denoted by row}$$

Solve the given equation by using Cramer's Rule

$$x + 2y + 3z = 6$$

$$3x - 2y + z = 2$$

$$4x + 2y + z = 7$$

Given system of Equations are

$$x + 2y + 3z = 6$$

$$3x - 2y + z = 2$$

$$4x + 2y + z = 7$$

Convert the given equation into augmented form $AX=B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$$

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

$$z = \frac{\Delta_3}{\Delta}$$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix}$$

$$= 1 [(-2)(1) - (1)(2)] - 2 [(3)(1) - (4)(1)] + 3 [(3)(2) - (-2)(4)]$$

$$= 1 (-2 - 2) - 2 (3 - 4) + 3 (6 + 8)$$

$$= 1 (-4) - 2 (-1) + 3 (14)$$

$$= -4 + 2 + 42$$

$$\Delta = 40$$

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 2 & -2 & 1 \\ 7 & 2 & 1 \end{vmatrix}$$

$$= 6 \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -2 \\ 7 & 2 \end{vmatrix}$$

$$= 6(-2 \times 2) - 2(2-7) + 3(4+14)$$

$$= 6(-4) - 2(-5) + 3(18)$$

$$= -24 + 10 + 54 = 40$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ 4 & 7 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 \\ 7 & 1 \end{vmatrix} - 6 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix}$$

$$= 1(2-7) - 6(3-4) + 3(21-8)$$

$$= 1(-5) - 6(-1) + 3(13)$$

$$= -5 + 6 + 39 = 40$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -2 & 2 \\ 4 & 2 & 7 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2 & 2 \\ 2 & 7 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix} + 6 \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix}$$

$$= 1(-14-4) - 2(21-8) + 6(6+8)$$

$$= 1(-18) - 2(13) + 6(14) = -18 - 26 + 84 = 40$$

$$x = \frac{\Delta_1}{\Delta} = \frac{40}{40} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{40}{40} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{40}{40} = 1$$

Verification:

$$x=1, y=1, z=1$$

1st equation $x+2y+3z=6$

$$1+2(1)+3(1)=6$$

$$1+2+3=6$$

$$6=6$$

Vectors: $\vec{r} + (r-r)\vec{B} - (r-r)\vec{C}$

$(r-r)\vec{C} + (r-r)\vec{B} - (r-r)\vec{C}$

$OP = r\vec{C} + r\vec{B} - r\vec{C}$

1. Find the eig. Eigen values and Eigen vectors of a matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol:

Given Matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Characteristic equation of Matrix A is given by $|A - \lambda I| = 0$ where λ is the Eigen value

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ -4 & 3-\lambda & \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -6 & -6 & -4 \\ 2 & 3-\lambda & \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -6 & 7-\lambda & -4 \\ 2 & -4 & \end{vmatrix} = 0$$

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = \Delta$$

$$(8-\lambda)(7-\lambda)(3-\lambda) - (-6)(-\lambda)(3-\lambda) - (2)(-\lambda)(-4) = 0$$

$$(8-\lambda)(7-\lambda)(3-\lambda) - (6\lambda)(3-\lambda) - (8\lambda) = 0$$

$$OP = r\vec{C} + r\vec{B} - r\vec{C}$$

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = \Delta$$

$$\Delta = 8(7)(3) - 6(6)(3) - 2(6)(4) = 168 - 108 - 48 = 12$$

$$\Delta = 12$$

$$\Delta = 12$$

$$\Delta = 12$$

$$\Delta = 12$$

$$\Delta = 12$$

$$\Delta = 12$$

initially

$$1 = 8, 1 = 7, 1 = 3$$

$$\vec{r} = 8\vec{C} + 7\vec{B} + 3\vec{C}$$

$$\vec{r} = 8\vec{C} + 7\vec{B} + 3\vec{C}$$

$$\vec{r} = 0$$

$$\begin{aligned}
 &\Rightarrow (8-\lambda) [(7-\lambda)(3-\lambda)-16] + 6[(1-6)(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0 \\
 &\Rightarrow (8-\lambda) [21-7\lambda-3\lambda+\lambda^2-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0 \\
 &\Rightarrow (8-\lambda) [\lambda^2-10\lambda+5] + 6[6\lambda-10] + 2[2\lambda+10] = 0 \\
 &\Rightarrow 8\lambda^2-8\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0 \\
 &\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0 \\
 &\Rightarrow -\lambda(\lambda^2-18\lambda+45)=0 \\
 &\Rightarrow \lambda=0, \lambda^2-18\lambda+45=0 \\
 &\quad \lambda^2-3\lambda-15\lambda+45=0 \\
 &\quad \lambda(\lambda-3)-15(\lambda-3)=0 \\
 &\quad (\lambda-3)(\lambda-15)=0 \\
 &\lambda=0; \lambda=3; \lambda=15
 \end{aligned}$$

$\therefore$ The Eigen values for the given Matrix $\lambda=0; \lambda=3; \lambda=15$
 To find Eigen vector for each Eigen values we use $(A-\lambda I)x=0$
 When $\lambda=0$ the Eigen vectors are

$$\begin{aligned}
 &(A-\lambda I)x=0 \\
 &\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0
 \end{aligned}$$

Substitute $\lambda=0$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 8x_1 - 6x_2 + 2x_3 \\ -6x_1 + 7x_2 - 4x_3 \\ 2x_1 - 4x_2 + 3x_3 \end{bmatrix} = 0$$

$$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0 \rightarrow \text{eq ①}$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \rightarrow \text{eq ②}$$

$$2x_1 - 4x_2 + 3x_3 = 0 \rightarrow \text{eq ③}$$

Solve any 2 equations by using numerical solving method to obtain x_1, x_2, x_3 values

Solve ① & ② $\Rightarrow (0-10)x + (0+10-10)y + (0-10+10)z = 0$ (1.8) $\leftarrow$

$-6x - 2y - 6z = 0$ (1.9) $\leftarrow$

$7x - 4y - 6z = 0$ (1.10) $\leftarrow$

$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$ $\leftarrow$

$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} = k_1$ $\leftarrow$

$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k_1$ $\leftarrow$

$x_1 = 1k_1$ $\leftarrow$

$x_2 = 2k_1$ $\leftarrow$

$x_3 = 2k_1$ $\leftarrow$

Now $\Rightarrow x = k_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\leftarrow$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 2k_1 \\ 2k_1 \end{bmatrix}$ $\leftarrow$

$\Rightarrow x = k_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\leftarrow$

$\therefore$ The Eigen vector then $\lambda=0$ is $x = k_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\leftarrow$

Substitute $\lambda=3$ $\leftarrow$

$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}$ $\leftarrow$

$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\leftarrow$

$\Rightarrow$ $\leftarrow$

$$5x_1 - 6x_2 + 2x_3 = 0 \rightarrow \text{eq ④}$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \rightarrow \text{eq ⑤}$$

$$2x_1 - 4x_2 + 0 = 0 \rightarrow \text{eq ⑥}$$

Solve any 2 Equations by using numerical solving method to obtain x_1, x_2, x_3 values

Solve ④ & ⑤

$$\begin{array}{rrrr} 5 & -6 & 2 & 5 \\ -6 & 4 & -4 & -6 \\ 2 & -4 & 0 & 2 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -24 & -8 & 20 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -16 & -8 & +16 \\ 2 & 1 & -2 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 2 & 1 & -2 \end{array}$$

$$x_1 = 2k_2$$

$$x_2 = 1k_2$$

$$x_3 = -2k_2$$

Now

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ 1k_2 \\ -2k_2 \end{bmatrix}$$

$$\Rightarrow x = k_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$\therefore$ The Eigen vectors then $\lambda = 3$ is $x = k_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Substitute $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ 6 & 7-15 & -4 \\ 2 & 4 & 3-15 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & 4 & -12 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & 4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0 \rightarrow \textcircled{5}$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \rightarrow \textcircled{6}$$

$$2x_1 + 4x_2 - 12x_3 = 0 \rightarrow \textcircled{7}$$

Solve any 2 Equations by using numerical solving method to obtain x_1, x_2, x_3 values

$$\begin{array}{r} -6 \quad 2 \quad -7 \quad -6 \\ -8 \quad -4 \quad -6 \quad -8 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 24+16 & -12-28 & 56-36 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 4/0_2 & -4/0_2 & 2/0_1 \end{array} = k_3$$

$$x_1 = 2k_3$$

$$x_2 = -2k_3$$

$$x_3 = 1k_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} k_3$$

$$\Rightarrow x = k_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vector where $\lambda = 15$ is $x = k_3 \times 2, -2, 1$

Vector definition:

Vector is an order of array of numbers it returns in either column or row. The numbers which constitute the vector are called components of a vector or elements of a vector.

Ex:

Geometrical and Physical interpretation of vectors:

Vectors are often used in many branches of physical and engineering science for quantitative, but have both magnitude (direction) and distance force and acceleration of a vectors.

Vectors are often represented ^{Physical array} geometrically by a line with an arrow.

The length of the line indicates the magnitude of the vector and it indicates the direction of slope.

A vector can also be represented as a point in the approximate dimension space.

UNIT-5 GAME THEORY

(Epimification)

Introduction:

In a competitive world, it is essential for an executive to study (i) at least guess the activities (ii) actions of his competitor.

Moreover, he has to plan his success for action (iii) reaction (iv) counter-action when his competitor uses several techniques.

Such as, war (i) game is regular feature in market and its competitor have to make this discuss in choosing their alternatives among the predictor customers. So, it has to maximum the profit (ii) minimize the losses.

Properties or Characteristics of Game theory.

There are various types of properties in game theory. they can be classified on the basis of following characteristics.

* Chance & strategy

→ In a game, activities are denoted by skills it is said to be game of strategies if they are determine by chance. In a game theory it is a game of a chance, in general game may involves game of strategy as well as game of chance.

② No. of Persons: (cricket - 11, Chess - 2, ...)

A Game is called an 'N'-persons game if the no. of persons playing one particular game is an individual (or) a group at that particular objectives.

*No of Activities: (eg. Betting)

This may be finite or infinite activities in a game theory. It is available to each person in a particular activities may also be finite (or) infinite. A finite persons has a finite no. of activities, no. of activities, each involving infinite numbers is said to be finite activities, otherwise the game is said to be infinite.

Information to the players.

Above the past activities of each player is completely available (or) partially available in game theory towards some pre-declared terms and conditions.

Pay off Matrix

A quantity measure of satisfaction. a person gets at the end of each game (or) play is called pay off matrix. It is a real value function of variables in the game theory.

Let, V_i is the pay off the players. i.e.,

$$V_i = 1 \leq i \leq n$$

In ~~and~~ an end person plays

if $\sum_{i=1}^n v_i = 0$ then the game is said to be zero-sum game.

Terminology of a game theory.

The terminology often used in the game theory is given below

* players : Each participant in a game is called a player.

* play

A play of the game is result when chooses a player has if he chooses course of action.

Two persons Zero-sum game.

There may be a two competitors playing a game if the number of players is two is called two person Zero-sum game. If the sum of the amount win by the all the winners is each to sum of the amount lose by the all the losers. A zero sum game (or) a constant game is the sum of gains & losses to zero. Suppose, a game has some gains (or) losses is not equal to zero then we call it as non-zero-sum game.

Strategies

A strategy of a player is the pre-determined which a player decides his course of action from his own list of the courses of an action during the game. strategies are two types they are:

* Pure strategy

* Mixed strategy

Pure strategy: eg (Closed tenders)

A pure strategy is a decision in advance of all players always to choose a particular course of the action. A pure strategy is usually represented by a number of which course of action is associated.

Mixed strategy: (eg open tenders)

A mixed strategy is a decision in advance of all the players to choose a course of action in some particular probability distribution this may be called as mixed strategy in some fixed position.

Two pers

Two person zero sum game (or) Rectangular game with Saddle point.

A Game said to be a two person zero sum game with two players or playing the game with a condition that the loss of one is the gain of other one.

If the player chooses only all strategy then the problem of process to find the saddle point and pure strategy exist in such game. This type of problems can be done by two ways

- * Saddle point mthd (or)
- * Maximum & Minimum mthd (or)
- * Domination mthd

Saddle point Mthd / Principle

It is evident from two person zero sum game that the row players tries to maximise profit and minimize loss on column players. i.e., applying the principles on column players tries to minimize the profit, maximize the losses i.e. applying the minimum

Principles of saddle point In a pay off matrix Maximum value in each row represents that minimum profit of that particular strategies. Similarly, the column players tries to maximise the profit & minimize the losses. on row players. i.e. applying the principles on row

players minimise the profit & maximise the losses i.e. applying the minimum principles of saddle point in a pay off matrix. Maximum value in each column represents that minimum profit of that particular strategy.

The players that choose the minimum value from the principle is called minimum of maximum principle. If the player chooses maximum value from the principle is called maximum of minimum principle.

If the maximum value of minimum rows and minimum value of maximum columns are equal then it is said to saddle point.

$$\text{i.e. } \max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$$

Solve the matrix using game theory to find

saddle point $\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix}$

Sol given Matrix $\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix}$

convert the given matrix into game theory

Player A $\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix}$

Saddle point $\Rightarrow \max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$

$\max \{ 1, -4, -1 \} = \min \{ 1, 5, 1 \}$

$1 = 1$

From the above saddle point statement,
 The player - A gains 1
 The player - B losses 1

$\therefore$ saddle point is 1
 Therefore, the game value = 1

Solve the game for the given matrix

$$\begin{bmatrix} -5 & -2 & 0 & 7 \\ 5 & -6 & -4 & 8 \\ 4 & 0 & 2 & 3 \end{bmatrix}$$

Given matrix

$$\begin{bmatrix} -5 & -2 & 0 & 7 \\ 5 & -6 & -4 & 8 \\ 4 & 0 & 2 & 3 \end{bmatrix}$$

Convert the given matrix into game theory

Player-A

$$\begin{matrix} & \text{Player-B} \\ \begin{bmatrix} -5 & -2 & 0 & 7 \\ 5 & -6 & -4 & 8 \\ 4 & 0 & 2 & 3 \end{bmatrix} \end{matrix}$$

Saddle point $\Rightarrow \max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$

$$\max \{-5, -6, 0\} = \min \{-5, -2, 0, 8\}$$

$$0 = 0$$

$\therefore$ The given matrix is zero sum game

Value of Game value = 0
 saddle point = 0

$\therefore$ It is a fair game.

Matrix:
$$\begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix}$$

$$\max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$$

$$\max \{ -1, -1, 6, -4 \} = \min \{ 16, 14, 6, 14, 12 \}$$

$$6 = 6$$

From the above saddle point statement

The said player 'A' gains = 6R

The player 'B' losses = 6R

Saddle point is 6

Value of game is 6

Game with mixed strategy

The problems of the game where the maximum value is not equal to min value then saddle point does not exist and are taken as mixed strategy it means both players will mixed different strategies with certain probability to optimised function.

A mixed strategy can be solved by following methods:

- * 2x2 game that algebraic method
- * 2xm & mx2 game of graphical game
- * nxm game is a linear programming method.

2x2 game that algebraic Method.

Consider 2x2 game with payoff matrix
 player A & player B. Algorithm to determine the mixed strategy as follows.

Step 1: find the absolute value of $a-b$ that can be written its against of Row-2

Step 2: find the absolute value of $c-d$ that can be written its against of Row-1

Step 3: find the absolute value of $a-d$ that can be written its against to Column-2

Step 4: find the absolute value of $b-c$ that can be written its against to Column-1.

Note: The absolute values are called addments. The payoff matrix will be used to find the value of game.

i.e.,

	a	b	
P-A	c	d	

$(b-d)$ $(a-c)$
 $(c-d)$ $(a-b)$
 $(b-c)$ $(a-d)$

addments

Formulas to solve for finding player.

$$\text{Player } A_1 = \frac{|d-c|}{|a-b| + |d-c|}$$

$$\text{Player } A_2 = \frac{|a-b|}{|a-b| + |d-c|}$$

$$\text{Player } B_1 = \frac{|d-b|}{|d-b| + |a-c|}$$

$$\text{Player } B_2 = \frac{|a-c|}{|d-b| + |a-c|}$$

Value of the game.

$$V_1 = \frac{(a \times |d-c|) + (c \times |a-b|)}{|d-c| + |a-b|} \quad (d)$$

$$V_2 = \frac{(b \times |d-c|) + (d \times |a-c|)}{|d-c| + |a-c|} \quad (d)$$

$$V_3 = \frac{(a \times |d-b|) + (b \times |a-c|)}{|d-c| + |a-c|} \quad (d)$$

$$V_4 = \frac{(c \times |d-b|) + (d \times |a-c|)}{|d-b| + |a-c|} \quad (d)$$

Another formula:

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

$$y_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

$$y_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

find the value of game for the following matrix with respect to a player-A & Player-B

$$\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix}$$

$$(V_1 = \frac{(a \times d - c \times b) + (c \times (a - b))}{(d - c) + (a - b)})$$

Given Matrix $\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix}$

Saddle point $\max\{\min \text{ row}\} = \min\{\max \text{ column}\}$

$$\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix} \quad \max\{6, 4\} = \min\{8, 9\}$$

$\therefore 6 \neq 8$ then saddle point is not satisfied
 hence, the given problem is mixed strategy

So, we need to find strategy for each player by using 2×2 strategy

$$\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix} \begin{matrix} (1-d) & d \\ (a-b) & 3 \end{matrix}$$

$$\frac{(6d) + (9(1-d))}{5} = \frac{(8(1-d)) + (4d)}{2}$$

$$\frac{6d + 9 - 9d + 9d}{5} = \frac{8 - 8d + 4d}{2}$$

$$\frac{6d + 9}{5} = \frac{8 - 4d}{2}$$

$$12d + 18 = 40 - 20d$$

$$32d = 22$$

$$d = \frac{11}{16}$$

from the above 2×2 game strategy we have

$$|a-b|=3$$

$$|c-d|=4$$

$$|a-c|=2$$

$$|b-d|=5$$

$$\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix}$$

Now, strategies for each player as follows

$$\text{Player } A_1 = \frac{|d-c|}{|a-b| + |d-c|} = \frac{4}{3+4} = \frac{4}{7} = 0.571$$

$$\text{Player } A_2 = \frac{|a-b|}{|a-b| + |d-c|} = \frac{3}{3+4} = \frac{3}{7} = 0.428$$

$$\text{Player } B_1 = \frac{|d-b|}{|d-b| + |a-c|} = \frac{5}{5+2} = \frac{5}{7} = 0.714$$

$$\text{Player } B_2 = \frac{|a-c|}{|d-b| + |a-c|} = \frac{2}{5+2} = \frac{2}{7} = 0.285$$

Value of game

$$V_2 = \frac{(b \times |d-c|) + (d \times |a-c|)}{|d-c| + |a-c|}$$

$$= \frac{(9 \times 4) + (4 \times 2)}{4 + 2} = \frac{36 + 8}{6} = \frac{44}{6} = 7.333$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{6 \times 4 - 9 \times 8}{6 - 9 - 8 + 4} = \frac{24 - 72}{-7} = \frac{-48}{-7} = 6.857$$

value for each player

player A = $[0.571, 0.428]$

player B = $[0.714, 0.285]$

find the saddle point and the game value for the matrix

Given matrix $\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$

saddle point $= \max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$
 $\max \{ 1, 2 \} = \min \{ 6, 7 \}$

$2 \neq 7$

$\therefore 2 \neq 7$, then saddle point is not satisfied.
 Hence the given problem is mixed strategy

So, we need to find strategy for each player by using 2×2 game strategy

$\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix} \begin{matrix} 1-d \\ 1-b \end{matrix} = 4$
 $\begin{matrix} b-d & a-c \end{matrix} = +6$

5 5

from the above 2×2 game strategy we have

$$|a-b| = 6$$

$$|c-d| = 4$$

$$|a-c| = 5$$

$$|b-d| = 5$$

Now strategy for each player is as follows

$$\text{Player } A_1 = \frac{|d-c|}{|a-b| + |d-c|} = \frac{4}{6+4} = 0.4$$

$$\text{Player } A_2 = \frac{|a-b|}{|a-b| + |d-c|} = \frac{6}{6+4} = 0.6$$

$$\text{Player } B_1 = \frac{|d-b|}{|d-b| + |a-c|} = \frac{5}{5+5} = 0.5$$

$$\text{Player } B_2 = \frac{|a-c|}{|d-b| + |a-c|} = \frac{5}{5+5} = 0.5$$

Other formula

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{2-6}{1-7-6+2} = \frac{-4}{-10} = 0.4$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{1-7}{1-7-6+2} = \frac{-6}{-10} = 0.6$$

$$y_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{2-7}{1-7-6+2} = \frac{-5}{-10} = 0.5$$

$$y_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{1-6}{1-7-6+2} = \frac{-5}{-10} = 0.5$$

$$\text{value} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}}$$

$$= \frac{2 - 42}{1 - 7 - 6 + 2} = \frac{-40}{-10} = 4$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}$$

saddle point $\max\{\min \text{ row}\} = \min\{\max \text{ column}\}$

$$\max\{1, 3\} = \min\{5, 4\}$$

$$3 \neq 4$$

It is a mixed strategy
 $\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}$ $L-d = 1$
 $a-b = 4$
 $a-c = 2$ $b-d = 3$

$$|a-b| = 4 \quad x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} = \frac{4-3}{5-1-3+4} = \frac{1}{5} = 0.2$$

$$|c-d| = 1$$

$$|b-d| = 3$$

$$|a-c| = 2$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} = \frac{5-1}{5-1-3+4} = \frac{4}{5} = 0.8$$

$$x_3 = \frac{a_{22} - a_{12}}{a_{11} - a_{21} - a_{12} + a_{22}} = \frac{4-1}{5-1-3+4} = \frac{3}{5} = 0.6$$

$$x_4 = \frac{a_{11} - a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} = \frac{5-3}{5-1-3+4} = \frac{2}{5} = 0.4$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{21} - a_{12} + a_{22}} = \frac{5 \times 4 - 3}{5-1-3+4} = \frac{20-3}{5} = \frac{17}{5} = 3.4$$

Dominance property in pure strategy and mixed strategy.

In some game it is possible to reduce the size of the pay matrix by eliminating rows (or) columns which are dominated by another row (or) column respectively.

Dominance property for rows ($x \leq y$).

If all the entries in a row should be less than or equal to the corresponding entries of another row then the row can be deleted.

Dominance property for columns ($x \geq y$).

If all the entries in a column should be greater than (or) equal to the corresponding entries of another column then the column can be deleted.

Problems in Pure Strategy

Solve the given game by using dominance method.

4	6	5	10	6
7	8	5	9	10
8	9	11	10	9
6	4	10	6	4

Solve Given game

	I	II	III	IV	V
Player A	4	6	5	10	6
	7	8	5	9	10
	8	9	11	10	9
	6	4	10	6	4

saddle point $\max\{\min \text{ row}\} = \min\{\max \text{ column}\}$

$$\max\{4, 5, 8, 4\} = \min\{8, 9, 11, 10, 10\}$$

$$8 \neq 8 = 8$$

value of game = 8

optimal solution using dominance method

Given matrix

Player-B

Player-A

$$\begin{matrix} & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \text{I} & 4 & 6 & 5 & 10 & 6 \\ \text{II} & 7 & 8 & 5 & 9 & 10 \\ \text{III} & 8 & 9 & 11 & 10 & 9 \\ \text{IV} & 6 & 4 & 10 & 6 & 4 \end{matrix}$$

Find the sum of each of row & each column of the given column game. &

identify $[x \leq y \text{ or } x \geq y]$, Player-B

Player-A

$$\begin{matrix} & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \\ \text{I} & 4 & 6 & 5 & 10 & 6 & 31 \\ \text{II} & 7 & 8 & 5 & 9 & 10 & 39 \\ \text{III} & 8 & 9 & 11 & 10 & 9 & 47 \\ \text{IV} & 6 & 4 & 10 & 6 & 4 & 30 \\ & 25 & 27 & 31 & 35 & 29 & x \leq y \end{matrix}$$

Player - A's strategy is dominated by player-

-A II strategy so, player A-II can be deleted.

	I	II	III	IV	V	
I	4	6	5	10	6	31
II	7	8	5	9	10	39
III	8	9	11	10	9	47
	↓	↓	↓	↓	↓	($x \geq y$)
Row	14	23	21	29	25	

player A-I strategy is dominated by player A-II

so, player A-I is deleted

	I	II	III	IV	V	
player A	7	8	5	9	10	39
II	8	9	11	10	9	47
	↓	↓	↓	↓	↓	($x \geq y$)
	15	17	16	19	19	

the row domination is not possible now
we choose column domination

player B-II strategy is dominated by

player B-I

Then B-II can be deleted

	I	II	III	IV	
	7	8	5	9	29
	8	9	11	10	38
	15	17	16	19	

player B-III is dominated by B-I so,

~~B-III~~ can

player A-III is dominated by player A-II

A-III can be deleted

player I $\begin{bmatrix} 5 & 9 & 11 & 10 \end{bmatrix}$

player B-III is dominated by B-I, delete B-III

player I $\begin{bmatrix} 5 & 9 & 10 \end{bmatrix}$

player B-III is dominated by B-I, delete B-III

delete B-II

player I $\begin{bmatrix} 5 & 9 \end{bmatrix}$

Two small columns - big

player B-I is dominated by B-II, so

delete B-I

player I $\begin{bmatrix} 5 \end{bmatrix}$

value of game is 5

strategy of player A $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

strategy of player B $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

solve the game by using to find optimal solution by using dominance property

$\begin{bmatrix} 5 & -3 & 3 & 4 \\ -4 & 5 & 4 & 5 \\ 4 & -4 & -3 & 3 \end{bmatrix}$

Given matrix $\begin{bmatrix} 5 & -3 & 3 & 4 \\ -4 & 5 & 4 & 5 \\ 4 & -4 & -3 & 3 \end{bmatrix}$

saddle point $\Rightarrow \max\{\min \text{ row}\} = \min\{\max \text{ column}\}$

$\max\{-3, -4, -4\} = \min\{5, 5, 4, 5\}$

$-3 \neq 4$

It is a mixed strategy. there is no saddle point. Now, we use dominance method to find optimal solution by using dominance method for value of game.

Given matrix

Player-B

Player-A $\begin{bmatrix} I & II & III & IV \\ I & 5 & -3 & 3 & 4 \\ II & -4 & 5 & 4 & 5 \\ III & 4 & -4 & -3 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & -2 & 4 & 12 \end{bmatrix} \begin{matrix} \rightarrow 9 \\ \rightarrow 10 \\ \rightarrow 0 \end{matrix}$

Player A-III strategy is dominated by A-I strategy

So, delete A-III

Resultant matrix

$\begin{bmatrix} I & II & III & IV \\ I & 5 & -3 & 3 & 4 \\ II & -4 & 5 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 7 & 9 \end{bmatrix} \begin{matrix} \rightarrow 9 \\ \rightarrow 16 \end{matrix}$

B-IV strategy is dominated by B-II strategy

Eliminate B-IV

Resultant game,

$$\begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} 5 \\ 5 \\ -4 \\ 4 \\ 1 \end{array} \begin{array}{c} 3 \\ -3 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} 3 \\ 3 \\ 4 \\ 4 \\ 3 \end{array} \begin{array}{c} -5 \\ 5 \\ 5 \\ 5 \\ 5 \end{array}$$

here there is no option that to check column domination (or) row domination. hence we can take any two columns for average.

$$\begin{bmatrix} \frac{5-3}{2} & 3 \\ \frac{-4+5}{2} & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2/2 & 3 \\ 1/2 & 4 \end{bmatrix}$$

$$\begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} 1 \\ 1 \\ 10.5 \end{array} \begin{array}{c} 3 \\ 3 \\ 4 \end{array}$$

player B-I is dominated player B-II hence.

we can eliminate B-2

$$\begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} 1 \\ 1 \\ 0.5 \end{array} \begin{array}{c} 3 \\ 3 \\ 4 \end{array}$$

~~Player A-I is dominated by player~~

The resultant value is Results of an average of 1st two columns in previous elimination. Now consider those columns for further considered

$$\Rightarrow \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -3 \\ -4 & 5 \end{bmatrix}$$

Now we have to find the value of the game by using 2×2 game method.

$$\begin{matrix} & a_{11} & a_{12} \\ a_{21} & \begin{bmatrix} 5 & -3 \\ a_{11} & a_{22} \\ -4 & 5 \end{bmatrix} \end{matrix}$$

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{5 + 4}{5 + 3 + 4 + 5} = \frac{9}{17}$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{5 + 3}{5 + 3 + 4 + 5} = \frac{8}{17}$$

$$y_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{5 + 3}{5 + 3 + 4 + 5} = \frac{8}{17}$$

$$y_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{5 + 4}{5 + 3 + 4 + 5} = \frac{9}{17}$$

$$\text{Value} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{25 - 12}{17} = \frac{13}{17}$$

~~Home work~~

$$\begin{bmatrix} 2 & 7 & 3 & 5 \\ 5 & 4 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 6 & 2 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 2 \\ 4 & 3 & 7 & -5 & 1 & 2 \\ 4 & 3 & 4 & -1 & 2 & 2 \\ 4 & 3 & 3 & -2 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

Given Matrix:
$$\begin{bmatrix} 2 & 7 & 3 & 5 \\ 5 & 4 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

Saddle point: $\max\{\min\text{row}\} = \min\{\max\text{column}\}$

$\max\{2, 4, 0\} = \min\{7, 7, 4, 5\}$

$4 = 4$

Value of game = 4

optimal solution by using dominance method

Given Matrix:
$$\begin{bmatrix} 2 & 7 & 3 & 5 \\ 5 & 4 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

Find the sum of each row & each column of the given game.

Player-A:
$$\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 2 & 7 & 3 & 5 \\ 5 & 4 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

Player-B:
$$\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{bmatrix} 2 & 7 & 3 & 5 \\ 5 & 4 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

Sum of rows: 14, 13, 7

Sum of columns: 17, 18, 12, 13

($17 \geq 18$)

Player-B-IV is dominated by B-3 eliminate

B-IV

Resultant matrix:
$$\begin{bmatrix} 2 & 7 & 3 \\ 5 & 4 & 4 \\ 7 & 2 & 0 \end{bmatrix}$$

Sum of rows: 12, 13, 9

Sum of columns: 12, 13, 7

Player B-1 is dominated by B-2 eliminate B-1

Resultant Matrix

$$\begin{array}{c}
 \begin{matrix} A \\ B \end{matrix} \begin{matrix} 1 & 2 \end{matrix} \\
 \begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 7 & 0 \end{bmatrix} \begin{matrix} \rightarrow 5 \\ \rightarrow 9 \\ \rightarrow 7 \end{matrix} \\
 \downarrow \quad \downarrow \\
 14 \quad 7
 \end{array}$$

A-1 is dominated by A-2 eliminate A-1

$$\begin{array}{c}
 \begin{matrix} A \\ B \end{matrix} \begin{matrix} 2 \end{matrix} \\
 \begin{bmatrix} 5 & 4 \\ 7 & 0 \end{bmatrix} \begin{matrix} \rightarrow 9 \\ \rightarrow 7 \end{matrix} \\
 \downarrow \quad \downarrow \\
 4 \quad 4
 \end{array}$$

B-2 is dominated by B-1 eliminate B-2

Resultant Matrix

$$\begin{array}{c}
 \begin{matrix} A \\ B \end{matrix} \begin{matrix} 1 \end{matrix} \\
 \begin{bmatrix} 4 \\ 0 \end{bmatrix}
 \end{array}$$

A-2 is dominated by A-1 eliminate A-2

Resultant Matrix

$$[4]$$

Value of game is 4, 0

Strategy of Player - A $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

Strategy of Player - B $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

Given Matrix $\begin{bmatrix} 5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix}$

saddle point $\max \{ \min \text{ row} \} = \min \{ \max \text{ column} \}$

$$\max \{ -5, -10, -20 \} = \min \{ 5, 10, 20 \}$$

$$-5 \neq 5$$

Mixed strategy

Given Matrix $\begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix} \begin{matrix} \rightarrow 25 \\ \rightarrow -15 \\ \rightarrow -35 \end{matrix}$

$\downarrow \quad \downarrow \quad \downarrow$
 $5 \quad -20 \quad -10$

Eliminate A-3 by comparing A-2

$$\begin{matrix} A & B \\ \text{I} & \begin{bmatrix} -5 & 10 & 20 \end{bmatrix} \rightarrow 25 \\ \text{II} & \begin{bmatrix} 5 & -10 & -10 \end{bmatrix} \rightarrow -15 \\ & \downarrow \quad \downarrow \quad \downarrow \\ & 0 \quad 0 \quad 10 \end{matrix}$$

Eliminate B-II

$$\begin{bmatrix} 5 & 10 \\ 5 & -10 \end{bmatrix} \begin{matrix} \rightarrow 5 \\ \rightarrow -5 \end{matrix}$$

$\downarrow \quad \downarrow$
 $0 \quad 0$

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{10 - 5}{-5 - 10 - 5 - 10} = \frac{-15}{-30} = \frac{1}{2}$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{5 - 10}{-5 - 10 - 5 - 10} = \frac{-15}{-30} = \frac{1}{2}$$